

A general geometric construction of coordinates in a convex simplicial polytope

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Abstract

Barycentric coordinates are a fundamental concept in computer graphics and geometric modeling. We extend the geometric construction of Floater's mean value coordinates [8,11] to a general form that is capable of constructing a family of coordinates in a convex 2D polygon, 3D triangular polyhedron, or a higher-dimensional simplicial polytope. This family unifies previously known coordinates, including Wachspress coordinates, mean value coordinates and discrete harmonic coordinates, in a simple geometric framework. Using the construction, we are able to create a new set of coordinates in 3D and higher dimensions and study its relation with known coordinates. We show that our general construction is complete, that is, the resulting family includes all possible coordinates in any convex simplicial polytope.

Key words: Barycentric coordinates, convex simplicial polytopes

1 Introduction

In computer graphics and geometric modelling, we often wish to express a point x as an affine combination of a given point set $v_\Sigma = \{v_1, \dots, v_i, \dots\}$,

$$x = \sum_{i \in \Sigma} b_i v_i, \quad \text{where} \quad \sum_{i \in \Sigma} b_i = 1. \quad (1)$$

Here $b_\Sigma = \{b_1, \dots, b_i, \dots\}$ are called the *coordinates* of x with respect to v_Σ (we shall use subscript Σ hereafter to denote a set). In particular, b_Σ are called *barycentric coordinates* if they are non-negative.

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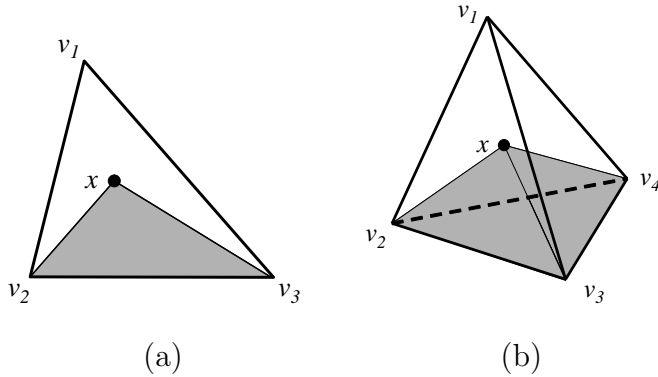


Fig. 1. Barycentric coordinates in a simplex.

In applications such as boundary value interpolation, one is interested in building coordinates when points v_Σ are the vertices of some closed polytope P and when x lies in the interior of P . In the simplest case, P is a non-degenerate simplex, such as a triangle in 2D or a tetrahedron in 3D. As such, the equation system of (1) has a unique solution with a simple geometric interpretation. As shown in Figure 1, b_i is the ratio of the volume of the simplex formed by x and the *opposite* face to vertex v_i over the volume of P . Such coordinates are also barycentric, that is, b_Σ are always positive when x lies inside the simplex P .

Unfortunately, for a general polytope P , the geometric construction in a simplex can no longer be used, because each vertex is no longer “opposite” to a particular face of P . To compound the situation, the under-determined equation system of (1) yields many solutions, and hence there may exist many sets of coordinates for a given point inside P .

One common approach to construct coordinates b_Σ is to look for weights w_Σ that satisfy

$$\sum_{i \in \Sigma} w_i (v_i - x) = 0. \quad (2)$$

Such weights are called *homogeneous* weights of x with respect to P . When the sum of the homogeneous weights is non-zero, normalizing w_i yields coordinates that satisfy equation (1). That is,

$$b_i = \frac{w_i}{\sum_i w_i}. \quad (3)$$

Observe that coordinates are homogeneous weights themselves. Therefore, a general construction that finds all sets of homogeneous weights in P is capable of constructing all sets of coordinates in P as well.

1.1 Previous work

Several sets of coordinates in 2D polygons and 3D polyhedra have been studied individually in the past, yet their constructions assume very different forms. *Wachspress coordinates* were first introduced by Wachspress [22] for convex 2D polygons, and was later extended to *weakly* convex polygons [14], convex polygons with internal nodes [15], and higher-dimensional convex polytopes [23]. Wachspress coordinates were defined in a non-trivial form involving ratios of volumes. Recently, a geometric interpretation was presented by Ju et. al. [12]. Given a convex polytope P and an interior point x , Wachspress weights are simply volumes of sub-polytopes of the polar dual of P with respect to x . *Mean value coordinates* were first introduced by Floater [6] for 2D polygons in a trigonometric form involving angles and edge lengths. Recently, extensions of mean value coordinates to 3D and higher dimensions were discovered by Floater et. al. [8] and Ju et. al. [11], who revealed the geometric interpretation of mean value weights as surface integrals over a unit sphere centered at x . Another class of coordinates for discrete shapes that has been widely used in mesh parameterization is the class of *discrete harmonic coordinates*. While discrete harmonic coordinates have a simple cotangent formula for 2D polygons [7], no geometric construction exists in the literature for 3D polyhedra or higher-dimensional polytopes. Among the three coordinates, both Wachspress and mean value coordinates are barycentric (e.g., positive) inside any convex shape, while discrete harmonic coordinates are positive only inside a 2D polygon whose vertices lie on a circle.

The difference in the construction of various coordinates obscures the relation among these coordinates and makes the discovery of new, meaningful coordinates difficult. To alleviate the problem, a unified construction was presented by Floater et. al. [7] in a convex 2D polygon. This construction involves ratios of triangle areas and a set of parameters, one associated with each vertex of the polygon, whose variation yields the entire family of coordinates in the polygon including Wachspress, mean value and discrete harmonic coordinates. However, to date, no general constructions are known for 3D polyhedra or higher dimensional polytopes.

1.2 Our contributions

Inspired by the geometric construction of mean value coordinates presented in [8,11], we introduce a general approach for building a family of coordinates in a piecewise linear shape. In 2D, our approach constructs coordinates in a convex polygon P by integration over an auxiliary, *generating curve*. Varying the shape of the generating curve yields a family of coordinates in P that includes Wachspress, mean value, and discrete harmonic coordinates. In particular, these coordinates correspond to generating curves with simple geometry, such

as the polar dual of P , the unit circle, and P itself.

Due to its geometric form, our general construction can be extended easily to 3D and higher dimensions. Given a convex triangular polyhedron P , we construct a family of 3D coordinates in P by varying the shape of a *generating surface*, yielding 3D Wachspress, mean value and discrete harmonic coordinates using the same set of simple shapes: the polar dual of P , the unit sphere and P itself. In addition, the construction allows intuitive creation of new 3D coordinates by designing a different generating surface. In particular, we consider a new set of 3D coordinates resulting from Voronoi duals and examine its relation to these known coordinates.

In general, our construction yields a family of coordinates in any convex simplicial polytope (a d -dimensional polytope P is simplicial if each facet of P is a $(d-1)$ -dimensional simplex.). This family unifies the three known coordinates in a simple, geometric manner, and makes it easy for discovery of meaningful coordinates that are naturally extended to all dimensions. A key theoretical result that we derive in this paper is that our construction is *complete*, in that the resulting coordinates family includes *all* possible coordinates in such polytopes. This is shown by relating our general construction to an equivalent form that directly extends Floater’s 2D general construction [7] to 3D and higher dimensions.

2 Constructing coordinates for convex 2D polygons

We first consider building coordinates in a convex 2D polygon, with the goal to derive a general construction that can be easily extended to higher dimensions. We start by reviewing the geometric construction of mean value coordinates presented in [8] and [11], which inspired our general approach.

2.1 Geometric construction of mean value coordinates

In a pioneering work [6], Floater introduced *mean value coordinates* in a polygon P with vertices v_Σ by defining a set of homogeneous weights w_Σ of the form

$$w_i = \frac{\tan[\alpha_{i-1}/2] + \tan[\alpha_i/2]}{|v_i - x|} \tag{4}$$

where α_i is the angle formed by $v_i - x$ and $v_{i+1} - x$. Normalizing w_i as in (3) yields coordinates of x with respect to P . Mean value coordinates enjoy additional properties such as being barycentric (i.e., positive) within any convex shape as well as within the kernel of any star shape. Hormann [10] further showed that mean value coordinates are continuously defined over arbitrary

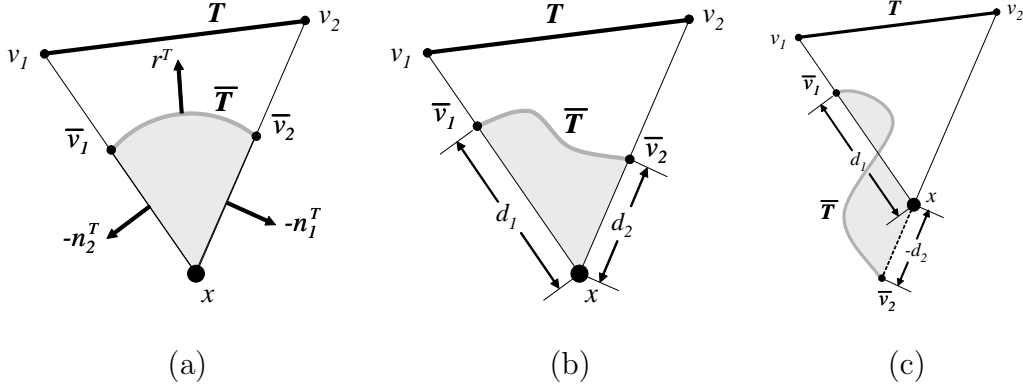


Fig. 2. Projecting an edge T onto a unit circle (a) and onto an arbitrary curve (b,c).

planar polygons.

Recently, Floater et. al. [8] and Ju et. al. [11] introduced a geometric construction of mean value coordinates, which allows extension of the coordinates into higher dimensions. They start by placing a unit circle centered at x , which intersects with rays emanating from x to each vertex v_i at points \bar{v}_i , as shown in Figure 2 (a). As such, an edge $T = \{v_1, v_2\}$ of the polygon is radially “projected” onto an arc \bar{T} on the circle with end points $\{\bar{v}_1, \bar{v}_2\}$.

We consider the integral of the outward unit normals over \bar{T} , and denote it as r^T . Since P is convex and x lies inside P , vectors $v_1 - x$ and $v_2 - x$ form a linearly independent basis, and r^T can be uniquely expressed in this basis as

$$r^T = u_1^T(v_1 - x) + u_2^T(v_2 - x). \quad (5)$$

The mean value weight of x with respect to a vertex v_i is constructed as the sum of the coefficient u_i^T in this expression for the two edges T containing v_i ,

$$w_i = \sum_{T: v_i \in T} u_i^T. \quad (6)$$

To verify that w_i constructed by equations (5) and (6) is the mean value weight defined in (4), we need an explicit representation of the integrals r^T . To this end, we consider the “wedge” enclosed by two straight segments $\{x, \bar{v}_1\}$, $\{x, \bar{v}_2\}$ and the circular arc \bar{T} , shown as the shaded region in Figure 2 (a). Due to Stokes’ Theorem [5], the integral of outward unit normals over any closed curve is zero. Since each straight segment $\{x, \bar{v}_i\}$ has unit length, applying the theorem to this wedge yields

$$r^T = n_1^T + n_2^T \quad (7)$$

where n_1^T, n_2^T are the *inward* unit normals of vectors $v_2 - x$ and $v_1 - x$, as shown in Figure 2 (a). On the other hand, solving (5) yields an explicit form

of the coefficients,¹

$$u_i^T = \frac{r^T \cdot n_i^T}{(v_i - x) \cdot n_i^T}. \quad (8)$$

Substituting (7,8) into (6) yields the trigonometric form of mean value weights in (4).

2.2 General construction of homogeneous coordinates

The geometric construction of mean value weights starts by projecting edges of the polygon onto arcs of a unit circle, and then represents the curve integral over each projected arc in the vector basis $v_i - x$. A natural generalization of this construction is to perform projection and integration on shapes other than the unit circle.

Consider a convex polygon P with vertices v_Σ and any closed curve G . Formally, we define a *projection* of P onto G with respect to x as a homeomorphic mapping from P to G such that the image \bar{v}_i of each vertex v_i under the mapping lies on the supporting line of $\{x, v_i\}$. Intuitively, this projection maps each edge $T = \{v_1, v_2\}$ on P to a curve segment \bar{T} on G with end points $\{\bar{v}_1, \bar{v}_2\}$, as shown in Figure 2 (b). G is called a *generating curve* if there exists such a projection of P onto G .

Note that a mapped vertex \bar{v}_i can be any one of the possibly multiple intersection points between the generating curve G and the supporting line of $\{x, v_i\}$, and it is not restricted to the half-line $\bar{x}\bar{v}_i$, as seen in the example of Figure 2 (c). As such, a generating curve G does not have to enclose x , and there may exist more than one projection of P onto G .

Let r^T in equation (5) denote the integral of the outward unit normals over a projected curve segment \bar{T} , we have:

Proposition 1 *Given x inside a convex polygon P , the weights w_Σ constructed by equations (5,6) are homogeneous for any generating curve G and a projection of P onto G with respect to x .*

Proof: Due to Stokes' theorem [5], the integral of outward unit normals over any closed curve G is zero. Since the set of all \bar{T} forms a partition of G , the sum of integrals r^T for all $T \in P$ is equally zero. Note that this sum can be re-written as a linear combination $\sum_i w_i(v_i - x)$ where w_i assumes the form of (5,6) (the expression of equation (5) is always unique since P is convex and x lies inside P). Hence the weights w_Σ satisfy the homogeneous condition of (2). \square

¹ Vector bases $v_i - x$ and $\frac{n_i^T}{(v_i - x) \cdot n_i^T}$ are known as reciprocal bases [1]

We refer to the construction in Proposition 1 as our *general construction* for 2D homogeneous weights (and coordinates). To compute the weights, we first explicitly express each integral r^T by applying Stokes' theorem to the wedge formed by $\{x, \bar{v}_1\}$, $\{x, \bar{v}_2\}$ and \bar{T} ,

$$r^T = d_1 n_2^T + d_2 n_1^T, \quad (9)$$

where $d_i = \frac{(\bar{v}_i - x) \cdot (v_i - x)}{|v_i - x|}$ is the signed magnitude of $\bar{v}_i - x$ along vector $v_i - x$, as shown in Figure 2 (b,c). Substituting r^T into equation (8) followed by (6) yields the weight w_i . Observe that when G is a unit circle, $d_i = 1$, and equation (9) reduces to (7), which results in mean value weights.

2.3 Example coordinates

By varying the shape of the generating curve, we can produce a family of sets of homogeneous weights. When the sum of weights in a set is non-zero, normalizing these weights yields coordinates as in equation (3). We have seen that this family of coordinates includes mean value coordinates when the generating curve is the unit circle centered at x . Here we examine another two members of this coordinates family that also correspond to generating curves with simple geometry.

Wachspress coordinates are one of the first barycentric coordinates introduced for convex shapes. In [11], the Wachspress coordinates are constructed geometrically via *polar duals*. The polar dual of a convex polygon P at point x is another convex polygon \hat{P}_x , where each vertex v_i of P corresponds to an edge of \hat{P}_x whose outward normal is $v_i - x$ and whose distance from x is $\frac{1}{|v_i - x|}$. The vertices of P and edges of \hat{P}_x are said to be *dual* to each other. Figure 4 left shows an example. The Wachspress weight w_i is simply the length of the dual edge of v_i on \hat{P}_x divided by $|v_i - x|$, or equivalently, the area of the triangle formed by x and the dual edge of v_i .

Discrete harmonic coordinates are coordinates that minimize the discrete harmonic functional over a triangulated domain. Given values f_i attached to each vertex v_i of a closed polygon P , and a value f_x attached to an interior point x , we can define a piecewise linear function $f[y]$ that interpolates the values at the vertices v_Σ and at x . The harmonic functional evaluates the integral

$$H = \int \int (\nabla f[y])^2 dP \quad (10)$$

where ∇ is the gradient operator. The choice of f_x that minimizes H can be expressed as an affine combination of f_Σ , where the affine coefficients are the discrete harmonic coordinates of x .

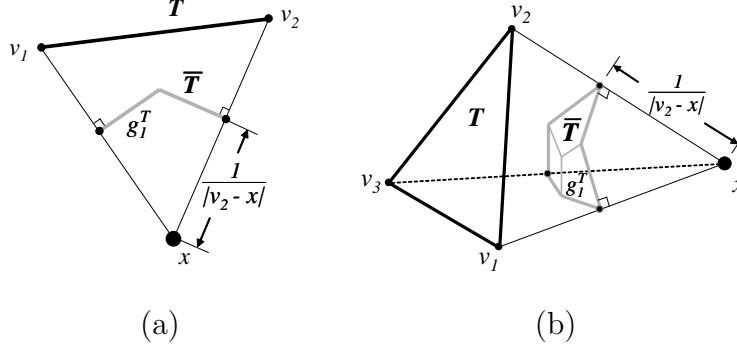


Fig. 3. Projecting a polygon edge T (a) or a polyhedral face T (b) onto the polar dual (colored gray) resulting in two linear segments (a) and three planar quads (b).

Using our general construction, we can relate these two coordinates in a geometric manner:

Proposition 2 *Given a convex polygon P and an interior point x , Wachspress coordinates and discrete harmonic coordinates can be constructed using generating curves \hat{P}_x and P .*

Proof: When $G = \hat{P}_x$, we consider the projection of P onto \hat{P}_x where $\bar{v}_i = x + \frac{v_i - x}{|v_i - x|^2}$. The projected curve segment \bar{T} of each edge $T = \{v_1, v_2\}$ hence consists of two straight segments g_1^T, g_2^T , each lying on the supporting line of the dual edge of v_1 or v_2 on \hat{P}_x , as shown in Figure 3 (a). Therefore, each coefficient w_i^T in the linear expression (5) computes the length of g_i^T divided by $|v_i - x|$. Since the sum of the lengths of g_i^T for the two edges T containing v_i is exactly the length of the dual edge of v_i , the weight w_i simply computes the length of the dual edge of v_i divided by $|v_i - x|$, which agrees with the formulation of Wachspress weights in [12].

When $G = P$, we consider the identity projection of P onto itself, that is, $\bar{v}_i = v_i$ and $\bar{T} = T = \{v_1, v_2\}$. Hence $r^T = |v_2 - v_1|n^T$ where n^T is the outward unit normal of T . For convenience, we use a different symbol for the un-normalized vector $|v_2 - v_1|n^T$ and denote it as N^T . Similarly, while n_1^T is the unit inward normal of edge $\{x, v_2\}$, N_1^T is the un-normalized vector $|v_2 - x|n_1^T$. Combining equations (6,8), we have

$$w_i = \sum_{T: v_i \in T} \frac{N^T \cdot N_i^T}{(v_i - x) \cdot N_i^T}. \quad (11)$$

On the other hand, when a point y lies in the triangle formed by x and edge T , the gradient $\nabla f[y]$ in (10) can be written as a linear expression of f_1 , f_2 and f_x ,

$$\nabla f[y] = \frac{N_1^T f_1 + N_2^T f_2 - N^T f_x}{2V[v_1, v_2, x]}, \quad (12)$$

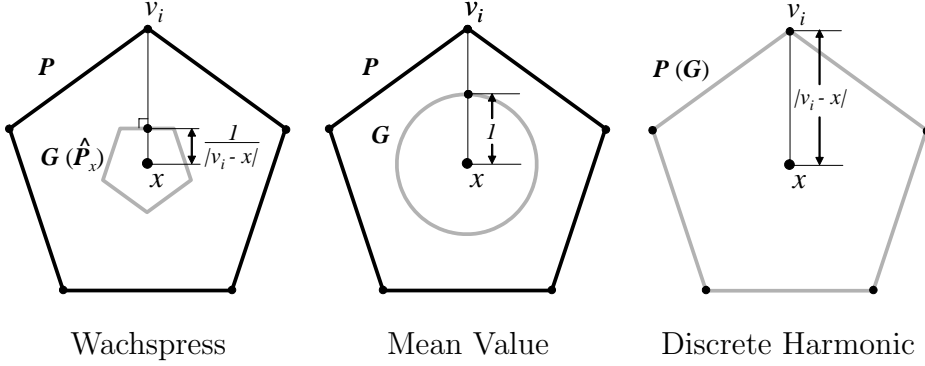


Fig. 4. Generating curves G (colored gray) in a convex polygon P : \hat{P}_x for Wachspress coordinates, unit circle for mean value coordinates, and P for discrete harmonic coordinates.

where $V[v_1, v_2, x]$ computes the signed area $\det[v_1 - x, v_2 - x]/2$. Solving $\frac{\partial H}{\partial f_x} = 0$ yields an expression of f_x as an affine combination $\sum w_i f_i / \sum w_i$, where each weight w_i has the form (11). \square

In Figure 4 we compare the generating curves that produce Wachspress, mean value and discrete harmonic coordinates. Note that the magnitudes d_i in equation (9) for computing each set of coordinates share a common form $|v_i - x|^\gamma$, where $\gamma = -1, 0, 1$ respectively for Wachspress, mean value and discrete harmonic coordinates.

Using 2D trigonometry, the formula we derived for discrete harmonic weights in (11) can be further simplified into a cotangent form

$$w_i = \cot[\beta_{i-1}] + \cot[\gamma_i] \quad (13)$$

where β_i is the angle formed by $v_i - x$ and $v_{i+1} - v_i$, and γ_i is the angle formed by $v_{i+1} - x$ and $v_{i+1} - v_i$, as shown in Figure 5. This formula has been observed and used in several previous works [19,4,7].

2.4 Relation to Floater's construction

Observe from equation (9) that the curve integrals r^T , and thus the resulting weights, are ultimately determined by magnitudes d_i that measure the distance from x to each projected point \bar{v}_i . In other words, the family of coordinates generated by our general construction are parameterized by these scalars associated with each vertex of the polygon. Such parameterization of coordinates is reminiscent of that proposed by Floater et. al. [7]. Specifically, let x be a point interior to a convex polygon P , and let each vertex v_i of the polygon be associated with some scalar c_i , Floater's construction computes

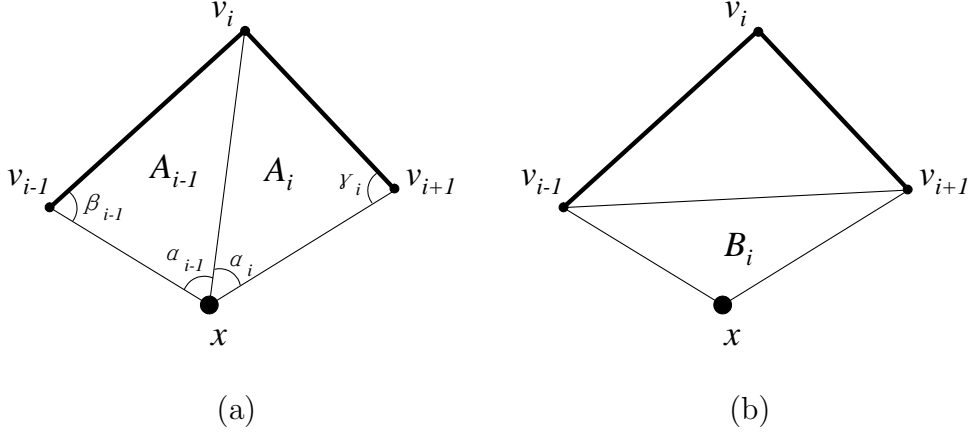


Fig. 5. Notations in Floater's construction.

the weight of x with respect to each vertex v_i as

$$w_i = \frac{c_{i-1}}{A_{i-1}} + \frac{c_{i+1}}{A_i} + \frac{c_i B_i}{A_{i-1} A_i} \quad (14)$$

where A_i and B_i are signed areas of triangles $\{x, v_{i-1}, v_i\}$ and $\{x, v_{i+1}, v_i\}$, as illustrated in Figure 5. Floater shows that the above weights are homogeneous for arbitrary sets of scalars c_Σ . In addition, equation (14) yields Wachspress, mean value and discrete harmonic coordinates when c_i is chosen respectively to be 1, $|v_i - x|$ and $|v_i - x|^2$.

Since both Floater's and our general construction depend on some scalar (c_i or d_i) associated with each vertex of the polygon, a natural question is whether the two constructions yield the same family of weights. This hypothesis is confirmed below:

Proposition 3 *Equations (6,8,9) and (14) compute the same set of weights w_Σ if $d_i = 2c_i/|v_i - x|$ for every vertex v_i .*

Proof: Using trigonometric expansion, the ratio $\frac{B_i}{A_{i-1} A_i}$ can be decomposed into a sum of two parts, each involving quantities within a triangle $\{x, v_{i-1}, v_i\}$ or $\{x, v_i, v_{i+1}\}$,

$$\frac{B_i}{A_{i-1} A_i} = -\frac{2 \cos[\alpha_{i-1}]}{l_i^2 \sin[\alpha_{i-1}]} - \frac{2 \cos[\alpha_i]}{l_i^2 \sin[\alpha_i]},$$

where α_i is the angle formed by $v_i - x$ and $v_{i+1} - x$, as shown in Figure 5 (a), and $l_i = |v_i - x|$. Hence equation (14) becomes

$$w_i = \frac{\frac{2c_{i-1}}{l_{i-1}} - \frac{2c_i}{l_i} \cos[\alpha_{i-1}]}{l_i \sin[\alpha_{i-1}]} + \frac{\frac{2c_{i+1}}{l_{i+1}} - \frac{2c_i}{l_i} \cos[\alpha_i]}{l_i \sin[\alpha_i]}. \quad (15)$$

On the other hand, substituting equations (8,9) into (6) yields a trigonometric form

$$w_i = \frac{d_{i-1} - d_i \cos[\alpha_{i-1}]}{l_i \sin[\alpha_{i-1}]} + \frac{d_{i+1} - d_i \cos[\alpha_i]}{l_i \sin[\alpha_i]}. \quad (16)$$

Equating (15) and (16) yields the relation $d_i = 2c_i/|v_i - x|$. \square

A key theoretical result in [7] is that construction (14) is *complete*, that is, any set of homogeneous weights (and therefore any set of coordinates) in a convex 2D polygon can be constructed using some set of c_Σ . As a result of the equivalence relation, and since any set of d_Σ in our construction corresponds to some generating curve G , we have shown that:

Corollary 1 *Any set of coordinates of x in a convex polygon P can be constructed using some generating curve G and some projection of P onto G with respect to x .*

3 Constructing coordinates for convex 3D triangular polyhedra

Our general, geometric construction in a 2D convex polygon can be easily extended to a 3D convex polyhedron with triangle faces. The 3D extension allows us to unify known 3D coordinates in a simple, geometric family. More importantly, we can now explore new 3D coordinates in an intuitive manner.

3.1 General construction in 3D

Let P be a convex triangular polyhedron with vertices v_Σ and edges $e_{\Sigma,\Sigma} = \{e_{i,j} | \{v_i, v_j\} \in P\}$. Given an interior point x , our construction closely follows our geometric approach in 2D. We define a *projection* of P onto G with respect to x as a homeomorphic mapping from P to G such that the image of each vertex v_i under the mapping, denoted as \bar{v}_i , lies on the supporting line of $\{x, v_i\}$, and the image of each edge $e_{i,j}$, denoted as $\bar{e}_{i,j}$, lies on the supporting plane of the triangle $\{x, v_i, v_j\}$. Intuitively, this projection maps each face $T = \{v_1, v_2, v_3\}$ on P to a surface patch \bar{T} on G enclosed by points $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ and curve segments $\{\bar{e}_{1,2}, \bar{e}_{2,3}, \bar{e}_{1,3}\}$, as shown in Figure 6 (a). As in 2D, G is called a *generating surface* if there exists such a projection of P onto G .

To construct the homogeneous weights given a generating surface G , we consider a triangle face T on P . Let r^T be the integral of the outward unit normals over the projected surface patch \bar{T} on G . Since P is convex and x lies inside P , vectors $v_1 - x$, $v_2 - x$, and $v_3 - x$ are linearly independent, hence form a

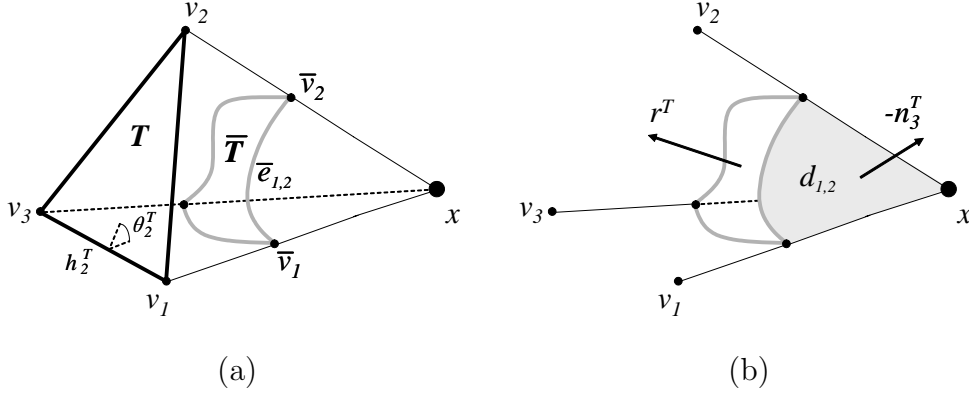


Fig. 6. Projecting a polygon T onto a generating surface (a), and computing the surface integral r^T (b).

basis in which r^T can be expressed using a unique set of coefficients,

$$r^T = \sum_{i=1}^3 u_i^T (v_i - x). \quad (17)$$

The weight w_i of x with respect to a vertex v_i is simply the sum of the coefficients u_i^T in this expression for all faces T containing v_i . The formulation of w_i is exactly the same as equation (6), with the only difference that now T represents a polyhedron face instead of a polygon edge. Using an argument analogous to the proof of Proposition 1, we can show that:

Proposition 4 *Given x inside a convex triangular polyhedron P , the weights w_Σ constructed by equations (17,6) are homogeneous for any generating surface G and a projection of P onto G with respect to x .*

Proof: The proof closely follows that of Proposition 1 by observing that the set of all \bar{T} forms a partition of G and that the integral of outward unit normals over any closed surface is zero (due to Stokes' theorem). \square

We refer to the construction in Proposition 4 as our *general construction* for 3D homogeneous weights (and coordinates). To compute the weights, we first express the integral r^T in an explicit form by applying Stokes' theorem to the "wedge" formed by three planar faces $\{x, \bar{v}_{i-1, i+1}\}$ ² and one surface patch \bar{T} ,

$$r^T = \sum_i^3 d_{i-1, i+1} n_i^T. \quad (18)$$

Here n_i^T is the inward unit normal of the oriented triangle³ $\{x, v_{i-1}, v_{i+1}\}$, and $d_{i-1, i+1}$ is the signed area enclosed by the oriented curve consisting of three segments, $\{x, \bar{v}_{i-1}\}$, $\bar{e}_{i-1, i+1}$ and $\{\bar{v}_{i+1}, x\}$. These notations are illustrated in

² The indices $\{i-1, i+1\}$ are cyclic, meaning $\{3, 2\}, \{1, 3\}, \{2, 1\}$ for $i = 1, 2, 3$.

³ We assume each triangle face of P is oriented to face outward.

Figure 6 (b). Substituting r^T into equations (8) followed by (6) yields the homogeneous weight w_i .

3.2 Example coordinates

3.2.1 Wachspress, mean value and discrete harmonic coordinates

Varying the generating surface G in our 3D general construction results in a family of sets of 3D homogeneous weights. If the sum of weights in a set is non-zero, these weights yield coordinates in a convex triangular polyhedron. In fact, when G is chosen as the unit sphere around x , our construction is exactly the same as the geometric construction for 3D mean value coordinates in [8] and [11]. Now we show that the other two types of coordinates constructed in 2D using simple generating curves can also be extended to 3D using simple generating surfaces:

Proposition 5 *Given a convex triangular polyhedron P and an interior point x , 3D Wachspress coordinates and 3D discrete harmonic coordinates can be constructed using generating surfaces \hat{P}_x and P .*

Proof: As in 2D, the polar dual \hat{P}_x is a convex polytope where each face is dual to a vertex v_i of P . In particular, the dual face of v_i on \hat{P}_x , denoted as \hat{T}_i , is orthogonal to $v_i - x$ and at a distance $\frac{1}{|v_i - x|}$ from x . When \hat{P}_x is the generating surface, we consider the projection of P onto \hat{P}_x where $\bar{v}_i = x + \frac{v_i - x}{|v_i - x|^2}$. As such, the projected surface patch \bar{T} of each face $T = \{v_1, v_2, v_3\}$ consists of three planar polygons g_i^T , each lying on the supporting plane of \hat{T}_i and being orthogonal to vector $v_i - x$ for $i = 1, 2, 3$, as shown in Figure 3 (b). Hence the area of g_i^T is measured by the coefficient u_i^T in (17) multiplied by $|v_i - x|$. Since the sum of areas of g_i^T for all faces T surrounding a vertex v_i is exactly area of the dual face \hat{T}_i , the weight w_i simply computes the area of \hat{T}_i divided by $|v_i - x|$, which agrees with the formulation of Wachspress weights in [12].

To derive 3D discrete harmonic coordinates, we first observe when a point y lies in the tetrahedron formed by x and triangle $T = \{v_1, v_2, v_3\}$, the 2D formula for gradient $\nabla f[y]$ in equation (12) can be extended to 3D as

$$\nabla f[y] = \frac{N_1^T f_1 + N_2^T f_2 + N_3^T f_3 - N^T f_x}{3V[v_1, v_2, v_3, x]} \quad (19)$$

where f_i and f_x are values at vertex v_i and x to be interpolated by $f[y]$, N^T is the outward normal of T with magnitude equal to the area of T , N_i^T is the inward normal of triangle $\{x, v_{i-1}, v_{i+1}\}$ with magnitude equal to the area of the triangle, and $V[v_1, v_2, v_3, x]$ computes the signed tetrahedral volume as $\det[v_1 - x, v_2 - x, v_3 - x]/6$. Interestingly, solving $\frac{\partial H}{\partial f_x} = 0$ yields f_x as an affine combination $\sum w_i f_i / \sum w_i$ where the discrete harmonic weights w_i have exactly the same form (11) as in 2D. On the other hand, when P is the generating

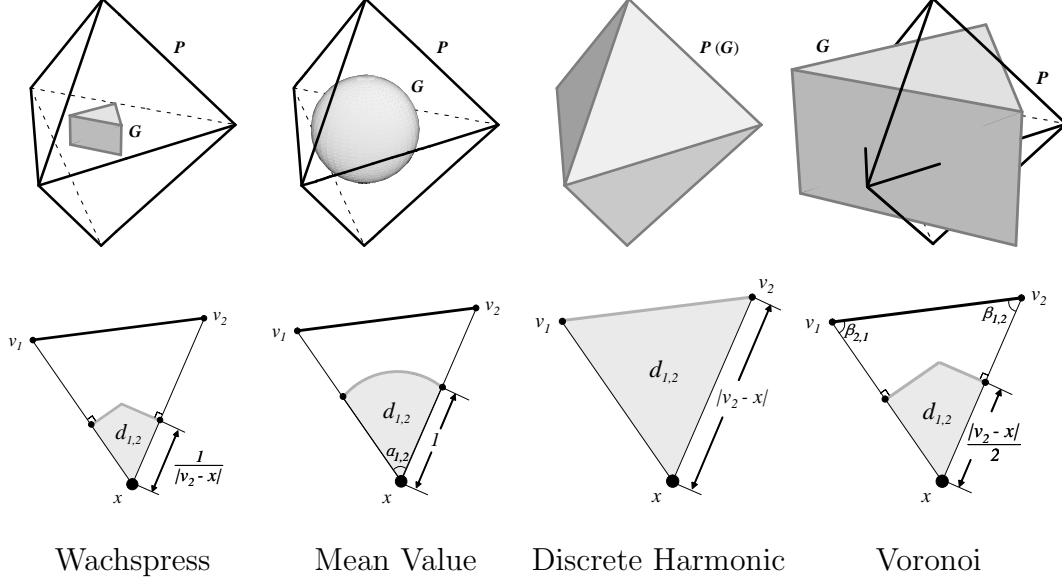


Fig. 7. Top: generating surfaces G in a convex triangular polyhedron P , showing the polar dual for Wachspress coordinates, the unit sphere for mean value coordinates, P for discrete harmonic coordinates, and the Voronoi dual for Voronoi coordinates. Bottom: scalars $d_{i,j}$ (areas of the shaded region) associated with each edge $\{v_i, v_j\}$.

surface, we consider the identity projection of P onto itself. Hence r^T is simply N^T , and combining equations (6,8) also yields w_i of the form (11). \square

The first three columns in Figure 7 compare the generating surfaces that produce Wachspress coordinates, mean value coordinates and discrete harmonic coordinates in 3D. To compute these coordinates using (18), the key is to determine the $d_{i,j}$ that measures the area of a 2D region within a closed curve formed by x and the projection of an edge $v_i v_j$ onto the generating surface. The bottom row of Figure 7 illustrates the corresponding 2D region for each generating surface that produces the three types of coordinates. Letting $\alpha_{i,j}$ denote the angle formed by $v_i - x$ and $v_j - x$, we compute: $d_{i,j} = \alpha_{i,j}/2$ for mean value coordinates; $d_{i,j} = |(v_i - x) \times (v_j - x)|/2$ for discrete harmonic coordinates; and

$$d_{i,j} = \frac{2 - \cos[\alpha_{i,j}] \left(\frac{|v_i - x|}{|v_j - x|} + \frac{|v_j - x|}{|v_i - x|} \right)}{2|(v_i - x) \times (v_j - x)|}$$

for Wachspress coordinates. Notice that in all three coordinates, $d_{i,j}$ involves only quantities within each triangle $\{x, v_i, v_j\}$. However, unlike scalars d_i in our 2D construction (or c_i in Floater's construction (14)), there is no obvious relation between $d_{i,j}$ for different 3D coordinates.

Using 3D trigonometry, the discrete harmonic weights in equation (11) can be further simplified into a cotangent form

$$w_i = \sum_{T: v_i \in T} \frac{\cot[\theta_i^T] h_i^T}{2} \quad (20)$$

where, within a triangle face $T = \{v_1, v_2, v_3\}$, θ_i^T is the dihedral angle between T and triangle $\{x, v_{i+1}, v_{i-1}\}$, and h_i^T is the edge length $|v_{i+1} - v_{i-1}|$, as shown in Figure 6 (a). The existence of cotangent weights in 3D was first mentioned in [18], without specifying an exact formula. Comparing with the 2D cotangent formula in equation (13), the 3D formula involves an additional edge length term.

3.2.2 Voronoi coordinates

Using our geometric construction, we can easily build new 3D coordinates by choosing other geometric shapes as the generating surface. Here we explore a special set of coordinates associated with Voronoi cells, known previously. As we shall see, this new set of coordinates is closely related to both Wachspress coordinates and discrete harmonic coordinates.

Given a point x and a point set v_Σ , the *Voronoi cell* of x is the set of all points in space that are closer to x than to any other points in v_Σ . Geometrically, the Voronoi cell is bounded by a convex polyhedron containing x . The vertices of the polyhedral boundary are circumcenters of some tetrahedra formed by points in $\{v_\Sigma, x\}$. One undesirable property of a Voronoi cell, however, is that the topology of its polyhedral boundary (e.g., number of vertices and faces) may vary with the location of x even when the v_i are fixed. To simplify coordinates computation, we would like to use a generating surface with a fixed topology structure (like the polar dual and the original polyhedron).

To this end, we introduce a variant of the Voronoi cell boundary. In 2D, for example, we define the *Voronoi dual*, \tilde{P}_x , of x with respect to a polygon P as another polygon that connects circumcenters of triangles formed by x and each edge of P . Figure 8 (a) shows an example. Observe that the Voronoi dual, \tilde{P}_x , has the same topology as the polar dual, \hat{P}_x (shown in Figure 8 (b) for comparison): each vertex (or edge) of \tilde{P}_x corresponds to an edge (or vertex) on P . In addition, each edge of \tilde{P}_x that corresponds to vertex v_i lies orthogonal to the vector $v_i - x$ with distance $|v_i - x|/2$ from x .

Similarly, in 3D, the Voronoi dual of x within a triangular polyhedron P is built from circumcenters of tetrahedra formed by x and each triangle face of P . Each face of \tilde{P}_x corresponds to a vertex v_i of P and connects circumcenters of tetrahedra formed by x and triangle faces containing v_i . An example of \tilde{P}_x as the generating surface is shown on the far right of Figure 7 top.

Using Voronoi duals as the generating curve (surface) and using the projection where each mapped vertex $\bar{v}_i = (x + v_i)/2$, our general construction produces a new type of 2D (3D) coordinates that we refer to as *Voronoi coordinates*. Note that Voronoi coordinates are extensions of the well-known Laplace coordinates, developed and improved by Hiyoshi and Sugihara [21,9], which are restricted to locations of x when the Voronoi dual \tilde{P}_x coincides with the Voronoi cell boundary of x (that is, when the Voronoi cell boundary consists of one face

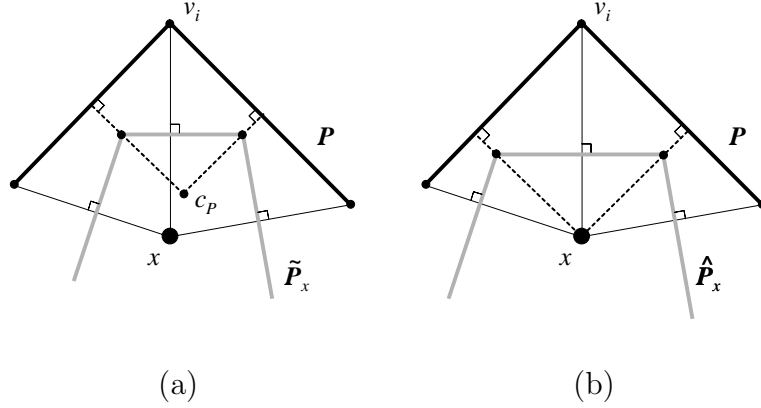


Fig. 8. Comparing a Voronoi dual (a) and a polar dual (b) (both colored gray) in a convex 2D polygon.

for every vertex of P).

Interestingly, Voronoi coordinates are already known in 2D by a different name:

Proposition 6 *Voronoi coordinates in a 2D convex polygon are identical to discrete harmonic coordinates.*

Proof: Observe that the scalars d_i in equation (9), which measure the length from x to each mapped vertex \bar{v}_i on the generating curve, are respectively $|v_i - x|/2$ and $|v_i - x|$ for the Voronoi dual and for the original polygon. After normalization of weights, the two sets of coordinates are identical. \square

Unfortunately, 3D Voronoi coordinates and 3D discrete harmonic coordinates are no longer identical. To compute 3D Voronoi coordinates, we observe from Figure 7 (bottom right) that $d_{i,j}$, used in equation (18), measures the area of a quadrilateral and has the form

$$d_{i,j} = \frac{1}{8}(|v_i - x|^2 \cot[\beta_{i,j}] + |v_j - x|^2 \cot[\beta_{j,i}])$$

where $\beta_{i,j}$ is the angle formed by $v_i - v_j$ and $x - v_j$. Note that this form is no longer a scaled version of $d_{i,j}$ for discrete harmonic coordinates (which is the area of the triangle $\{x, v_i, v_j\}$). In general, these two coordinates result in different values in a convex polyhedron (see next section for an example).

An even more interesting property of Voronoi coordinates is their relation to Wachspress coordinates in both 2D and 3D:

Proposition 7 *Wachspress coordinates and Voronoi coordinates are identical within a convex polygon (or convex triangular polyhedron) whose vertices lie on a circle (or sphere).*

Proof: The key to proving the identity is to show that the Voronoi dual and the polar dual are *similar* within such P . First, consider a 2D polygon P whose

vertices lie on a circle centered at c_P . Since vertices of the Voronoi dual, \tilde{P}_x , are circumcenters of triangles formed by x and edges of P , vectors emanating from c_P to vertices of \tilde{P}_x are on the perpendicular bisectors of edges of P (see Figure 8 (a)). On the other hand, by definition of the polar dual, vectors emanating from x to vertices of \hat{P}_x are normal to these edges too (see Figure 8 (b)). In addition, the edges on \tilde{P}_x and \hat{P}_x that correspond to the same vertex v_i on P are both normal to $v_i - x$. As a result, \tilde{P}_x and \hat{P}_x are similar, that is, corresponding angles are equal and corresponding edges are in proportion.

Similarly, consider a 3D polyhedron P with circumcenter c_P . The vector emanating from c_P to a vertex of \tilde{P}_x is parallel to the vector emanating from x to the corresponding vertex of \hat{P}_x (both orthogonal to some face of P), and corresponding faces of \tilde{P}_x and \hat{P}_x are parallel (both normal to some vector $v_i - x$). Hence \tilde{P}_x and \hat{P}_x are similar.

Using a proof analogous to that of Proposition 2 and 5, we can show that a Voronoi weight w_i , like a Wachspress weight, can be interpreted as the size of the edge (or face) of \tilde{P}_x that corresponds to vertex v_i on P divided by $|v_i - x|$. Since \tilde{P}_x and \hat{P}_x are similar, Voronoi weights and Wachspress weights are in proportion, and the resulting coordinates are identical due to normalization. \square

Floater et. al. [7] first discovered that discrete harmonic coordinates are identical to Wachspress coordinates in a 2D polygon whose vertices lie on a circle, which agrees with our findings here. Furthermore, the above results seem to indicate that equivalence with Wachspress is not an intrinsic property of discrete harmonic coordinates, but rather a property of the Voronoi coordinates.

3.2.3 Examples

In Figures 9 and 10 we compare the four types of 3D coordinates with respect to a particular vertex of a polyhedron. The coordinates are plotted at cross-sections of the polyhedron, using the grayscale to indicate larger (lighter) or smaller (darker) values. In addition, negative coordinates are drawn in black. Observe that while Wachspress coordinates and mean value coordinates are barycentric (i.e., positive), discrete harmonic coordinates are generally not, even when the vertices of the polyhedron lie on a sphere (such as the octahedron in Figure 9).

Observe from Figure 9 that Voronoi coordinates are exactly the same as Wachspress coordinates in a polyhedron whose vertices lie on a sphere. However, the two coordinates are generally different in other polyhedra, as shown in Figure 10. In addition, Voronoi coordinates are not always barycentric. Figure 11 shows an example of a convex polyhedron where Voronoi coordinates become negative.

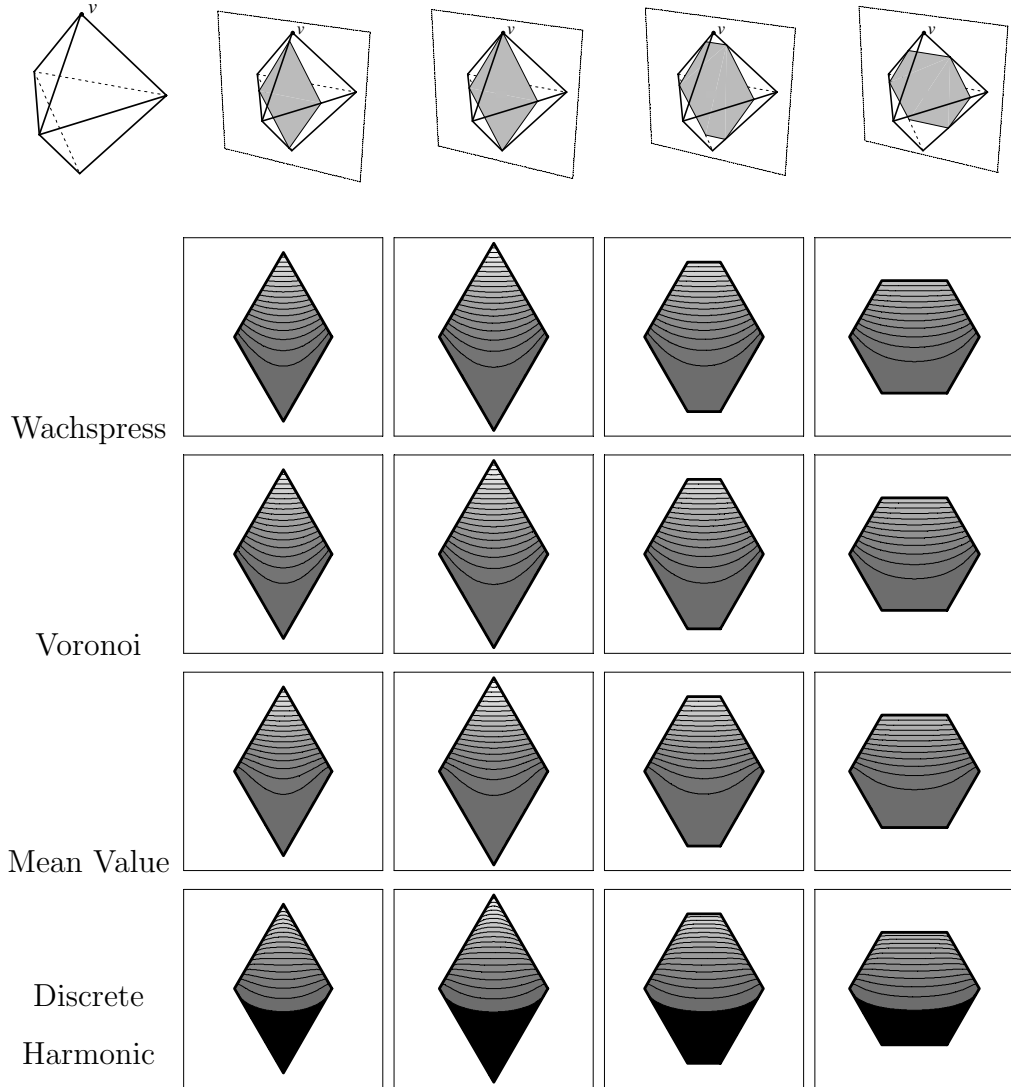


Fig. 9. Plotting four types of coordinates with respect to vertex v in an octahedron whose vertices lie on a sphere. Each coordinate is plotted on cross-sections of the polyhedron shown at the top. Lighter colors indicate larger coordinate values, and black regions indicate negative coordinates.

3.3 Extending Floater's construction to 3D

In this section we will show an equivalent form of our general construction in 3D that closely resembles Floater's 2D construction in (14). The extended Floater's construction in 3D will allow us to show the completeness of our general construction in the next section.

Like our construction, the 3D extension of Floater's construction produces a family of coordinates by varying parameters associated with each *edge* of the

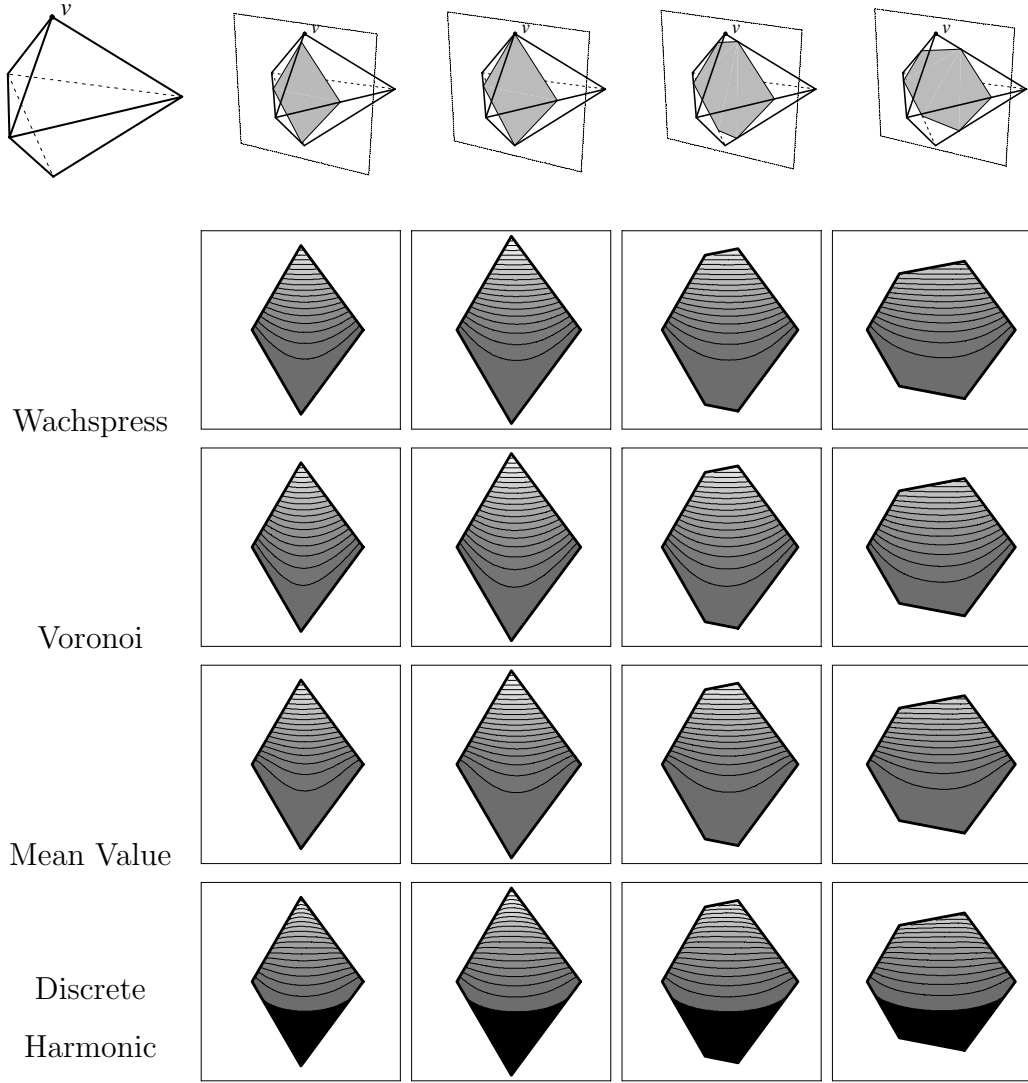


Fig. 10. Plotting four types of coordinates with respect to vertex v in an octahedron whose vertices do not lie on a sphere. Each coordinate is plotted on cross-sections of the polyhedron shown at the top. Lighter colors indicate larger coordinate values, and black regions indicate negative coordinates.

polyhedron. Consider a vertex v_i on a triangular polyhedron P , and let the vertices in the one-ring neighborhood of v_i have consecutive indices enumerated by j . We construct the weight w_i as

$$w_i = \sum_j \frac{c_{j,j+1}}{A_j} + \sum_j \frac{c_{i,j} B_j}{A_{j-1} A_j} \quad (21)$$

where $c_{i,j}$ is an arbitrary scalar, and A_j, B_j are respectively the signed volumes $V[v_i, v_j, v_{j+1}, x]$ and $V[v_{j-1}, v_{j+1}, v_j, x]$. The notations are illustrated in in Fig-

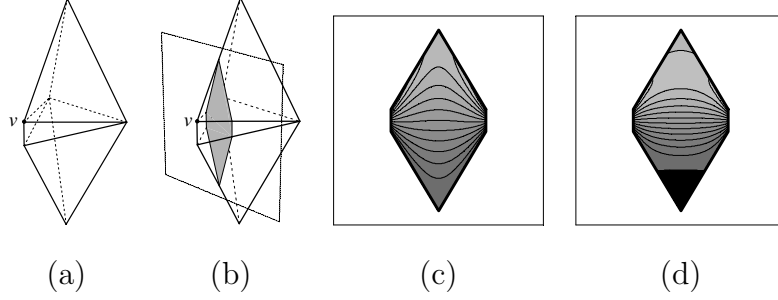


Fig. 11. Comparing the Wachspress (c) and Voronoi (d) coordinates with respect to vertex v in a convex polyhedron (a). The coordinate is plotted on a cross-section of the polyhedron shown in (b). Black region in (d) indicates negative values.

ure 12. Note that this formula, which is essentially a sum of volume ratios, is analogous to Floater's 2D formula (14), which is a sum of area ratios. Using a proof analogous to that in [7], we can show that the weights computed by (21) are homogeneous:

Proposition 8 *Let x be a point inside a convex triangular polyhedron P . Equation (21) computes homogeneous weights for any set of scalars $c_{\Sigma, \Sigma} = \{c_{i,j} | \{v_i, v_j\} \in P\}$ associated with edges of P .*

Proof: We first consider a vertex v_i and its three consecutive neighboring vertices v_{j-1} , v_j and v_{j+1} , as shown in Figure 12(a). Computing the barycentric coordinates of x with respect to the tetrahedron $\{v_i, v_{j-1}, v_j, v_{j+1}\}$ yields

$$A_j(v_{j-1} - x) + A_{j-1}(v_{j+1} - x) + B_j(v_i - x) + C_j(v_j - x) = 0 \quad (22)$$

where C_j is the signed volume $V[v_{j-1}, v_i, v_{j+1}, x]$, as shown in Figure 12(c). We can divide both sides of equation (22) by the product $A_{j-1}A_j$ and get

$$\frac{(v_{j-1} - x)}{A_{j-1}} + \frac{(v_{j+1} - x)}{A_j} + \frac{B_j(v_i - x)}{A_{j-1}A_j} + \frac{C_j(v_j - x)}{A_{j-1}A_j} = 0. \quad (23)$$

Note that we can write one identity (23) for each edge $\{v_i, v_j\}$ on P . Therefore, if we denote the left-hand side of (23) as $D_{i,j}$, and associate each edge $\{v_i, v_j\}$ with a scalar $c_{i,j}$, the following sum is identically zero:

$$\sum_{\{v_i, v_j\} \in P} c_{i,j} D_{i,j} = 0. \quad (24)$$

The coefficient of each vector $v_i - x$ in the left-hand side of the equation (24) is the weight w_i computed by (21). Hence w_i satisfies the homogeneous conditions in (2). \square

Note that both (21) and our general construction are parameterized by scalars ($c_{i,j}$ or $d_{i,j}$) associated with each edge of the polyhedron. As in 2D, we can

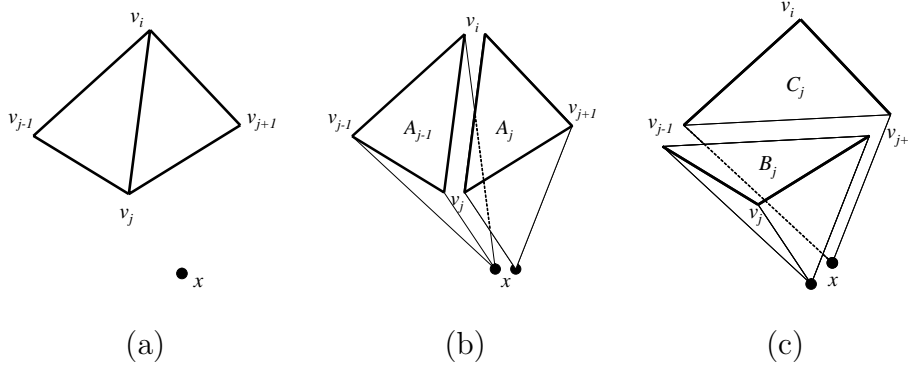


Fig. 12. Notations in a 3D polyhedron.

establish their equivalence:

Proposition 9 *Equations (6,8,18) and (21) compute the same set of weights w_Σ if*

$$d_{i,j} = \frac{6c_{i,j}}{|(v_i - x) \times (v_j - x)|} \quad (25)$$

for every edge $\{v_i, v_j\}$.

Proof: First we observe a trigonometric form of tetrahedral volume. Denoting the area of triangle $\{x, v_i, v_j\}$ as $S_{i,j}$, the volume of A_j in Figure 12 can be computed as

$$A_j = \frac{2}{3|v_j - x|} S_{i,j} S_{j+1,j} \sin[\theta_{i,j,j+1}] \quad (26)$$

where $\theta_{i,j,j+1}$ is the dihedral angle formed by triangles $\{x, v_j, v_i\}$ and $\{x, v_j, v_{j+1}\}$. As in 2D, using trigonometric expansion, the ratio $\frac{B_j}{A_{j-1}A_j}$ can be decomposed into a sum of two parts, each involving quantities within a tetrahedron $\{x, v_{j-1}, v_j, v_i\}$ or $\{x, v_j, v_{j+1}, v_i\}$,

$$\frac{B_j}{A_{j-1}A_j} = -\frac{\cos[\theta_{j-1,j,i}]S_{j-1,j}}{S_{i,j}A_{j-1}} - \frac{\cos[\theta_{i,j,j+1}]S_{j+1,j}}{S_{i,j}A_j}.$$

Hence equation 14 becomes a summation over all triangles around v_i ,

$$w_i = \sum_j \frac{\frac{c_{j,j+1}[x]}{S_{j,j+1}} - \frac{c_{i,j}[x]}{S_{i,j}} \cos[\theta_{i,j,j+1}] - \frac{c_{i,j+1}[x]}{S_{i,j+1}} \cos[\theta_{j,j+1,i}]}{A_j/S_{j,j+1}}. \quad (27)$$

Substituting equations (18) and (8) into (6) and equating with (27) yields the relation (25). \square

3.4 Proof of completeness

As a key theoretical contribution of this paper, we prove in this section that our general construction in 3D is complete. That is, the construction is capable of expressing any set of homogeneous weights (or coordinates) in a convex 3D triangular polyhedron. This is shown by proving the completeness of the extended Floater's construction (21), which is equivalent to our general construction. The proof is inspired by the completeness argument in [7] for Floater's original construction in 2D:

Proposition 10 *Given a convex triangular polyhedron P with vertices v_Σ and an interior point x , any set of homogenous weights w_Σ of x can be constructed by (21) using some set of scalars $c_{\Sigma,\Sigma} = \{c_{i,j} | \{v_i, v_j\} \in P\}$.*

Proof: Let P consist of n vertices and m edges. Without loss of generality, we assume that vertices $\{v_{n-2}, v_{n-1}, v_n\}$ form a triangle on the polyhedron. We can re-write the first $n - 3$ equations in (21) in matrix form as

$$M \cdot C = \{w_1, \dots, w_{n-3}\}^T \quad (28)$$

where C is a column vector of m unknown scalars $c_{\Sigma,\Sigma}$, one for each edge, and M is an $(n - 3)$ by m sparse matrix involving ratios of volumes.

First we show that a solution to (28) exists, or equivalently, the rows of matrix M are linearly independent. The key observation is that the k th column of M has non-zero entries only at rows $i, j, j - 1$ and $j + 1$, where edge $\{v_i, v_j\}$ is the k th edge on the polyhedron and the indexing is as shown in Figure 12 (a). Therefore, given an arbitrary non-zero vector $a = \{a_1, \dots, a_{n-3}\}$ and letting $a \cdot M = b = \{b_1, \dots, b_m\}$, we get

$$b_k = a_i M_{i,k} + a_j M_{j,k} + a_{j-1} M_{j-1,k} + a_{j+1} M_{j+1,k}. \quad (29)$$

Furthermore, we let $V = \{v_t | a_t \neq 0\}$ be a subset of all the vertices on the polyhedron. Note that $V \neq \emptyset$ and $V \cap \{v_{n-2}, v_{n-1}, v_n\} = \emptyset$. Since all the triangles on the polyhedron form a single connected component, there exist two adjacent triangles, say $T_1 = \{v_i, v_j, v_{j-1}\}$ and $T_2 = \{v_i, v_j, v_{j+1}\}$, such that $T_1 \cap V \neq \emptyset$ and $T_2 \cap V = \emptyset$. Hence the right-hand side of (29) reduces to a single, non-zero product $a_{j-1} M_{j-1,k}$. Therefore, $b \neq 0$, and the row vectors in M are linearly independent.

Given a solution set $c_{\Sigma,\Sigma}$ to equation (28), we can use the last three equations of (21) to construct weights \tilde{w}_{n-2} , \tilde{w}_{n-1} and \tilde{w}_n . Due to Proposition 8 and the homogeneous condition (2), we have the identity

$$\sum_{i=n-2}^n \tilde{w}_i(v_i - x) = - \sum_{i=1}^{n-3} w_i(v_i - x) = \sum_{i=n-2}^n w_i(v_i - x)$$

Observing that vectors $(v_i - x)$ are linearly independent, we have $\tilde{w}_i = w_i$ for $i = n - 2, n - 1$ and n . This completes the proof, as we have constructed a set of scalars $c_{\Sigma, \Sigma}$ that give rise to the given weights w_{Σ} . \square

Following the equivalence relation established by Proposition 9, and since any set of scalars $d_{\Sigma, \Sigma}$ in our construction corresponds to some generating surface, we have shown that:

Corollary 2 *Any set of coordinates of x in a convex triangular polyhedron P can be constructed using some generating surface G and some projection of P onto G with respect to x .*

4 Remarks on higher dimensions

Our general construction can be extended to higher dimensions without difficulty. Given an N -dimensional convex simplicial polytope P with vertices v_i and an interior point x , we can construct weights w_i by first projecting each $(N - 1)$ -dimensional facet of P onto some closed *generating hyper-surface* G , followed by representing the surface integral over each projected patch in the vector basis $v_i - x$. The resulting weights have the form (6) and are homogeneous given any such G .

As in 2D and 3D, higher-dimensional counterparts of Wachspress, mean value and discrete harmonic coordinates can be produced in our general construction using the polar dual, unit hyper-sphere and the original polytope as the generating hyper-surfaces. In addition, using the Voronoi dual as the generating hyper-surface would result in coordinates that are equivalent to Wachspress coordinates within a convex polytope whose vertices lie on a hyper-sphere.

Finally, it can be shown that Floater's construction in higher dimensions is formulated as a sum of volume ratios similar to the 3D formula and is equivalent to our general geometric construction. Like our construction, the extended Floater's construction generates a family of coordinates parameterized by scalars associated with each $(N - 2)$ -dimensional facet of P . Furthermore, the proof of Proposition 10 can be extended to any convex simplicial polytopes to show that both constructions are capable of expressing any coordinates in those polytopes.

5 Conclusion and discussion

We have presented a general construction of coordinates for convex 2D polygons, 3D triangular polyhedra, and higher-dimensional simplicial polytopes. The construction is built upon a generating shape (curve, surface, etc.), and varying the geometry of the generating shape produces a family of coordinates.

We showed that Wachspress coordinates, mean value coordinates and discrete harmonic coordinates are members of this family corresponding to simple generating shapes that are consistent in all dimensions. We also presented a new set of coordinates using the Voronoi dual as the generating shape. Finally, we extended Floater’s 2D construction to 3D and higher dimensions. We showed that the two constructions are equivalent, and that the family of coordinates produced by either construction includes all possible coordinates in a convex simplicial polytope.

To conclude, we briefly discuss a different interpretation of our coordinates construction and present some open problems for future research.

5.1 Gradient interpretation

Desbrun *et. al.* [4] showed that 2D discrete harmonic weights, when restricted to an edge T of the polygon, are related to the gradient of the area of the triangle formed by $\{x, T\}$ evaluated at x . Here we use our geometric construction to show that any set of homogeneous weights in an N -dimensional polytope P is related to the gradient of some volume.

Let T be a facet of P . Given x , we consider a generating hyper-surface G and the projected surface patch \bar{T} on G . Let’s further consider any point y interior to G , and the gradient of the volume of the solid “wedge” formed by $\{y, \bar{T}\}$ as y varies in G . We will show that such gradient evaluated at $y = x$ is directly related to the homogeneous weights at x restricted to T . The key observation is that such gradient is in fact a constant multiple of r^T , the integral of outward unit normals over \bar{T} . This is because the product of the area of an infinitesimal patch on \bar{T} and the outward unit normal at that patch is proportional to the gradient of the volume of any “cone”, whose base is the infinitesimal patch, as the tip of the cone varies. On the other hand, as shown in equations (5,17), r^T can be expressed as a linear combination of vectors $v_i - x$, where the linear coefficients u_i^T form the homogeneous weights at x restricted to T .

The above gradient interpretation agrees with Desbrun’s observation of discrete harmonic weights in a 2D polygon P [4], since the generating curve G for discrete harmonic weights is P itself (and hence $\bar{T} = T$).

5.2 Open problems

Our study of coordinate construction is by no means complete. Many questions remain open, especially in finding coordinates with specific properties and in building coordinates for shapes other than convex simplicial polytopes.

First, we would like to find coordinates besides Wachspress coordinates and mean value coordinates that are barycentric (i.e., positive) within any convex

polytope. Floater *et. al.* [7] gave two examples of such coordinates in a 2D polygon by designing appropriate scalar functions c_i for each vertex v_i . It would be interesting to see examples of such coordinates in 3D using the extended construction presented here, and to ask whether choices of these scalar functions correspond to generating curves (or surfaces) with particular shape properties. In addition, we are also interested in characterizing the class of generating curves (or surfaces) that result in homogeneous weights with non-zero sum, so that coordinates are well-defined by normalization.

Second, we would like to extend the general construction to non-convex polytopes. The main difficulty can be illustrated in 2D: a point x interior to a non-convex polygon P may lie on the supporting line of an edge of P , hence the basis decomposition in equation (5) may not exist. Visually, the resulting coordinates may exhibit “poles” inside the polygon, as demonstrated in [11]. Although it has been shown [10] that mean value coordinates are well-defined over arbitrary polygons, the challenge remains in identifying other coordinates constructed using the general approach here that have such properties. It would also be interesting to cast several other interpolants [16,17,3] developed recently for non-convex polygons into our geometric framework and determine their corresponding generating curves (or surfaces).

Third, we have only considered polytopes with simplicial facets. Given a 3D polyhedron with a non-triangular planar face $T = \{v_1, \dots, v_k\}$ where $k > 3$, the decomposition of the integral r^T into the vector basis $v_i - x$ for $i = 1, \dots, k$ becomes non-unique. Although any decomposition results in homogeneous weights, the challenge lies in finding the right decomposition so that the coordinates have the desired behavior. For example, Wachspress coordinates, as in [12], are in fact defined over both simplicial and non-simplicial polytopes. Also, mean value coordinates are derived from mean value interpolation [11] that applies to any continuous surface. Hence the task is to find a decomposition scheme that agrees with Wachspress coordinates or mean value interpolation over non-simplicial shapes. Recently Langer *et. al.* [13] presented a decomposition scheme for non-triangular faces using spherical barycentric coordinates, and showed that the resulting 3D coordinates converge to 2D coordinates of the same type as x moves from interior of P onto a planar face of P . However, it remains open if Langer’s scheme results in Wachspress coordinates or mean value interpolation in such polyhedra.

Last but not least, while both Wachspress coordinates and mean value coordinates have been extended to continuous shapes [24,11], the “link” between continuous coordinates and discrete coordinates remains unclear in general. As the first step towards bridging the gap, Schaefer *et. al.* [20] presented a unified coordinate construction over continuous 2D curves that is capable of expressing all discrete coordinates when the curve is piece-wise linear. Recently, Belyaev [2] presented a general coordinate construction over continuous 3D surfaces that is capable of producing several known coordinates in the con-

tinuous domain. With the general coordinate construction for discrete shapes developed in this paper, the natural question is how Belyaev’s method is related to our method when applied to piece-wise linear surfaces. Such study should shed light into developing a unified construction of coordinates over both continuous and discrete shapes in any dimension.

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