A Proof of Arrow's Theorem

The following proof is used in Austen-Smith and Banks's Positive Political Theory, Volume I. The actual proof of the theorem relies on a lemma, which we will prove first. The lemma requires the following three definitions:

Definition 1 *A coalition* $L \subseteq N$ *is* **semidecisive for (x, y)** *if* $[x \succ_i y$ *for all* $i \in L$ *and* $y \succ_j x$ *for all* $j \notin L] \Rightarrow x \succ y$.

Definition 2 *A coalition* $L \subseteq N$ *is* **decisive for (x, y)** *if* $x \succ_i y$ *for all* $i \in L \Rightarrow x \succ y$.

Definition 3 *A coalition* $L \subseteq N$ *is* **decisive** *if, for all* $(x, y) \in X \times X$, $x \succ_i y$ *for all* $i \in L \Rightarrow x \succ y$.

Examples of decisiveness:

- Player i is a dictator. Then $i \in L \Rightarrow L$ decisive.
- Majority rule. Then $|L| > \frac{|N|}{2} \Rightarrow L$ decisive.

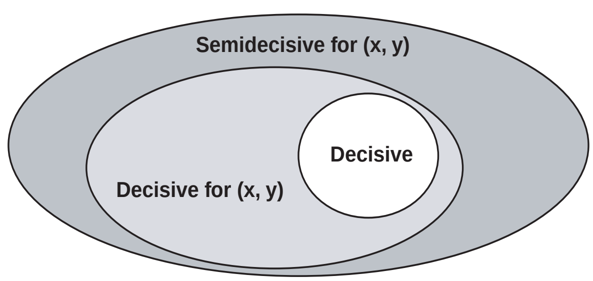

Figure 1: Note than *decisive* $\Rightarrow$ *decisive for (x, y)* $\Rightarrow$ *semidecisive for (x, y)*.

Lemma 1 *Let social welfare function F be quasitransitive, IIA, and Pareto. Then L semidecisive for (x, y) implies that L is decisive.*

Proof: Recall the definition of quasitransitivity: $x \succ y$ and $y \succ z \Rightarrow x \succ z$. Assume that coalition L is semidecisive for (x, y). We will show that L is also decisive.

Let preference profile ρ be such that for all $i \in L$, $x \succ_i z$.

Now consider a different profile ρ' such that:

For all $i \in L$, $x \succ_i y \succ_i z$, and

for all $j \notin L$, $y \succ_j x$ and $y \succ_j z$.

Then under profile ρ' we know that:

$x \succ y$ because L semidecisive over (x, y)

$y \succ z$ by the fact that F satisfies Pareto

$x \succ z$ by quasitransitivity.

Therefore, under our original ρ we must have that $x \succ z$ by the assumption of IIA. This is the trickiest part of this proof. It follows because we can assume that the players' (x, z) rankings are unchanged between ρ and ρ'. All members of L prefer x to z under ρ and ρ'. However, preferences over x and z are unspecified for all $j \notin L$ under both ρ and ρ'. Therefore we can let them be whatever we want, and so we can, without loss of generality, assume that they are the same under ρ and ρ'.

Since we have $x \succ z$ under ρ, the following condition (*) follows:

(*) For all $z \notin \{x, y\}$, L semidecisive for (x, y) $\Rightarrow$ L decisive for (x, z).

Since decisive implies semidecisive we know that L is also semidecisive for (x, z).

Switching z and y we get from (*) that L is decisive for (x, y).

Now let ρ^o be any profile such that $y \succ z$ for all $i \in L$.

Let ρ^* be a different profile such that:

For all $i \in L$, $y \succ_i x \succ_i z$.

For all $j \notin L$, $z \succ_j x$ and $y \succ_j x$.

By (*) we know that L is decisive for (x, z), and so under ρ^* we get:

$x \succ z$ because L is decisive for (x, z)

$y \succ x$ because F satisfies Pareto

$y \succ z$ by quasitransitivity.

By the previous argument (IIA) it follows that $y \succ z$ under our original ρ^o. We now get condition (**):

(**) For all $z \notin \{x, y\}$, L semidecisive for (x, y) $\Rightarrow$ L is decisive for (y, z).

Similarly, switching z and x we get that L is decisive for (y, x).

Finally, combining (*) and (**) we get that for *any* distinct pair $\{v, w\} \subseteq X \setminus \{x, y\}$,

L semidecisive for (x, y) implies:

L decisive for (x, v) by (*)

L decisive for (v, w) by (**), replacing y with v.

It follows that for all ordered pairs $(x, y) \in X \times X$, L is decisive for (x, y). Thus, L is decisive.

□

Theorem 1 *(Arrow) Consider any social welfare function F which always produces a transitive ordering over 3 or more alternatives. Assume at least two voters, and that every voter's preferences form a strict order and satisfy universal domain. Then if F satisfies Binary Independence and Pareto, F is a dictator.*

Proof: Using the previous lemma, it suffices to show that there exists an $i \in N$ such that i is semidecisive over some pair $(x, y) \in X \times X$.

The Pareto assumption implies that N is decisive, and thus, semidecisive over every pair.

For every ordered pair $(a, b) \in X \times X$, let $\lambda(a, b)$ be the size of the smallest semidecisive group for (a, b). Let λ be the smallest $\lambda(a, b)$, so that $\lambda = min_{(a,b)\in X\times X}\lambda(a, b)$.

Suppose L is semidecisive for (x, y), with $|L| = \lambda$. If $\lambda = 1$, our proof is complete. Assume that $\lambda > 1$.

Consider a profile ρ such that:

For one $i \in L$, $x \succ_i y \succ_i z$

For all $j \in L \setminus \{i\}$, $z \succ_j x \succ_j y$

For all $k \notin L$, $y \succ_k z \succ_k x$.

Then under ρ we get that:

$x \succ y$ because L is semidecisive for (x, y)

$y \succeq z$ because $L \setminus \{i\}$ is *not* semidecisive

$x \succ z$ by transitivity.

However, this implies that i is semidecisive for (x, z), by IIA. By the previous lemma this implies that Player i is decisive, and is therefore a dictator.

□