Fair Division in Theory and Practice

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Lecture 8: Simple rules and the Nakamura number

Last week and today we will think about voting systems as producing families of *decisive coalitions*

The structure of the family of decisive coalitions a voting rule generates can tell us about certain properties of the rule

(this lecture focuses mostly on transitivity as such a property)

Decisive coalitions

- A group of people, $L \subseteq N$ is $decisive$ if whenever $x \succ_i y$ for all i in L, then socially $x \succ y$.
- 3 people and majority rule generate the following decisive coalitions: $\{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$
- L denotes a particular decisive coalition, and $\mathcal{L}(f)$ denotes the $family$ of decisive coalitions induced by the aggregation rule f (e.g. majority rule).

Examples

Think about a weighted voting system in which everyone has a (possibly different) voting weight, v_i. If the sum of a group's weight is greater then $q = .5$ then that group is decisive. Find the families of decisive coalitions for the following weighted voting systems:

- $v_1 = .51, v_2 = .49$
- $v_1 = .4, v_2 = .1, v_3 = .5$
- $v_1 = .3, v_2 = .4, v_3 = .05, v_4 = .15, v_5 = .1$

Examples

Think about a weighted voting system in which everyone has a (possibly different) voting weight, v_i. If the sum of a group's weight is greater then .5 then that group is decisive. Find the families of decisive coalitions for the following weighted voting systems:

- $v_1 = .51, v_2 = .49$: $\{\{1\}, \{1, 2\}\}$
- $v_1 = .4, v_2 = .1, v_3 = .5$: $\{\{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
- $v_1 = .3, v_2 = .4, v_3 = .05, v_4 = .15, v_5 = .1$: $\{\{1, 2\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4\}\}$ (minimally decisive)

Families of decisive coalitions are always *monotonic* and *proper*

- Monotonic: if $L \in \mathcal{L}(f)$ and $L \subseteq L'$, then $L' \in \mathcal{L}(f)$
  - If L is a decisive coalition, then any set containing L is also a decisive coalition
- If $L \in \mathcal{L}(f)$ then $N \setminus L \notin \mathcal{L}(f)$
  - If L is a decisive coalition, then the people *not* in L cannot constitute a decisive coalition

Simple rules

A simple aggregation rule is *completely characterized* by its family of decisive coalitions

- With a simple rule we only need to know the family of decisive coalitions in order to know how the rule itself works
- Simple rules are the ONLY rules that satisfy this property

Formal definition:

An aggregation rule f is a *simple rule* if:

$$x \succ y \underline{\textbf{ if and only if }} \text{there exists an } L \in \mathcal{L}(f) \text{ with}$$

$$x \succ_i y \text{ for all } i \in L$$

If there is no decisive coalition L that unanimously strictly prefers x to y or y to x, then it must be the case that $x \sim y$ in the social ranking.

Plurality rule: $x \succ y$ if and only if the number of people that strictly prefer x to y is greater than the number of people that strictly prefer y to x.

What is the family of decisive coalitions?

Is plurality rule a simple rule?

Plurality rule is not simple

With three people, $\mathcal{L}(f) = \{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$

Consider the following profile:

$$x \succ_1 y \succ_1 z$$

$$x \sim_2 y \sim_2 z$$

$$x \sim_3 y \sim_3 z$$

Social ranking is $x \succ y \succ z$, but there is not a decisive coalition that prefers x to y, for example

With three people, $\mathcal{L}(f) = \{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$

What is the simple rule that is defined by this family of decisive coalitions (which was plurality rule's family of decisive coalitions)?

What ranking would this simple rule produce when given the last profile:

$$x \succ_1 y \succ_1 z$$

$$x \sim_2 y \sim_2 z$$

$$x \sim_3 y \sim_3 z$$

Suppose we have the following family of (minimal) decisive coalitions: $\{\{1,3\},\{1,2\}\}$

Any family of decisive coalitions will define a simple rule.

Given the simple rule defined by this family of coalitions, how would it aggregate the following preference profile:

$$x \succ_1 y \succ_1 z$$

$$x \succ_2 y \sim_2 z$$

$$z \succ_3 y \succ_3 x$$

Suppose we have the following family of (minimal) decisive coalitions: $\{\{1,3\},\{1,2\}\}$

Given the simple rule defined by this family of coalitions, how would it aggregate the following preference profile:

$$x \succ_1 y \succ_1 z$$

$$x \sim_2 z \succ_2 y$$

$$z \succ_3 y \succ_3 x$$

Suppose we have the following family of (minimal) decisive coalitions: $\{\{2,3\},\{1,2\}\}$

Given the simple rule defined by this family of coalitions, how would it aggregate the following preference profile:

$$x \succ_1 y \succ_1 z$$

$$x \sim_2 z \succ_2 y$$

$$z \succ_3 y \succ_3 x$$

Is Borda count with three individuals and three alternatives a simple rule? (What is its family of decisive coalitions?)

No. There is only one decisive coalition: $\{\{1,2,3\}\}$

$$x \succ_1 y \succ_1 z$$

$$z \succ_2 x \succ_2 y$$

$$x \succ_3 y \succ_3 z$$

Social ranking is $x \succ y \sim z$, but coalition $\{1,3\}$ has preferences $y \succ z$. At the same time, $\{1,3\}$ decides the x, z ranking, against the preferences of 2. Therefore, Borda outcomes are not completely characterized by its family of decisive coalitions.

Last example (for now):

Think about a weighted voting rule where a coalition is decisive if its voting weights sum to greater than or equal to 0.6.

What is the family of minimal decisive coalitions when $v_1 = .2, v_2 = .4, v_3 = .3, v_4 = .2$?

How does the simple rule described by the family aggregate the following preferences?

$$x \succ_1 y \succ_1 z$$

$$z \succ_2 x \succ_2 y$$

$$x \succ_3 z \succ_3 y$$

$$x \succ_4 y \succ_4 z$$

Think about a weighted voting rule where a coalition is decisive if its voting weights sum to greater than or equal to 0.6.

What is the family of minimal decisive coalitions when $v_1 = .2, v_2 = .4, v_3 = .3, v_4 = .2$? $\{\{1,2\},\{1,3,4\},\{2,3\},\{2,4\}\}$

How does the simple rule described by the family aggregate the following preferences?

$$x \succ_1 y \succ_1 z$$

$$z \succ_2 x \succ_2 y$$

$$x \succ_3 z \succ_3 y$$

$$x \succ_4 y \succ_4 z$$

$$x \succ z \succ y$$

The collegium

The collegium $K(\mathcal{L}(f))$ is defined as follows:

$$K(\mathcal{L}(f)) = \cap_{L_i \in \mathcal{L}(f)} L_i$$

In words, $K(\mathcal{L}(f))$ is the intersection of all of the decisive coalitions within our family of decisive coalitions, $\mathcal{L}(f)$.

If $K(\mathcal{L}(f)) \neq \emptyset$, then the rule f is called *collegial*.

Find the collegia of the following families of decisive coalitions:

- $\{\{2, 3\}, \{1, 2\}\}$
- $\{\{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 5, 6\}\}$
- Majority rule with three people

The *Nakamura number*, $s(f)$, associated with a simple aggregation rule f equals ∞ if f is collegial.

Otherwise:

$$s(f) = \text{Min}\{|\mathcal{L}| : \mathcal{L} \subseteq \mathcal{L}(f) \text{ and } K(\mathcal{L}) = \emptyset\}.$$

In words, $s(f)$ is the size of the smallest collection of decisive coalitions produced by the rule that have an empty intersection.

Examples of the Nakamura number, $s(f)$:

Majority rule with three people:
$\mathcal{L}(f) = \{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$, $s(f) = 3$.

Majority rule when $N = \{1,2,3,4\}$?
Majority rule when $N = \{1,2,3,4,5\}$?
For unanimity rule?
For a $\frac{4}{5}$ voting rule with 5 people?

We can use the Nakamura number to look at the size of a policy space that a procedure can accommodate while still being incapable of producing cycles.

First, how big of a policy space (i.e. how many alternatives) does majority rule need to produce a cycle?

Theorem: A simple rule, f, is incapable of producing cycles if and only if $|X| < s(f)$.

In words, only if the size of the policy space is strictly less than the Nakamura number can we guarantee that cycles can't happen.

What does this tell us about majority rule?

About a simple rule that is collegial (i.e. has a person or collection of people in every decisive coalition)?

UN Security Council prior to 1965

11 total members, with 5 permanent members (China, France, Great Britain, Soviet Union, U.S.A.)

To pass a motion 7 total "yes" votes were required, along with the concurring votes of all 5 permanent members. Taking "yes" and "concurring" to mean that $x \succ_i y$, what are the decisive coalitions produced by this rule?

Could this rule produce cyclic outcomes?

The Nakamura number provides a measure of the “rationality” of an aggregation procedure: the larger the Nakamura number, the larger a policy space our procedure can cope with, in terms of being able to produce “best” outcomes.

Interestingly, in multidimensional policy spaces, the Nakamura number provides a bound on the dimensionality of the policy space, k, needed in order to guarantee the existence of a nonempty core.

Theorem: When f is simple (and preferences are “single-peaked”) then $k \leq s(f) - 2$ implies the core is nonempty.

What does this say about majority rule?

The Spatial Model

- Policy space is no longer finite, and now equals a subset of k-dimensional Euclidean space
  - All of the points on a one-dimensional interval, like $[0, 1]$
  - All the points on a subset of a 2-dimensional plane, such as $[0, 1] \times [0, 1]$

The Pareto set for a group L

A policy x is Pareto-dominated for a group $L \subseteq N$ if there is some other policy y such that $every$ member of L strictly prefers y to x.

For any group L, the Pareto set for L is the set of policies that are not Pareto dominated for L.

Suppose this picture shows the *ideal points* of 10 voters in a one-dimensional policy space.

- What is the Pareto set for $L = \{3, 7\}$?
- For $L = \{2, 3, 4\}$?
- For $L = N$?

Theorem: For any simple rule, the core (in a spatial voting context) is the intersection of the Pareto sets of all decisive coalitions in $\mathcal{L}(f)$.

The core is the set of policies that are undefeated by any other policy (the set of "best" policies)

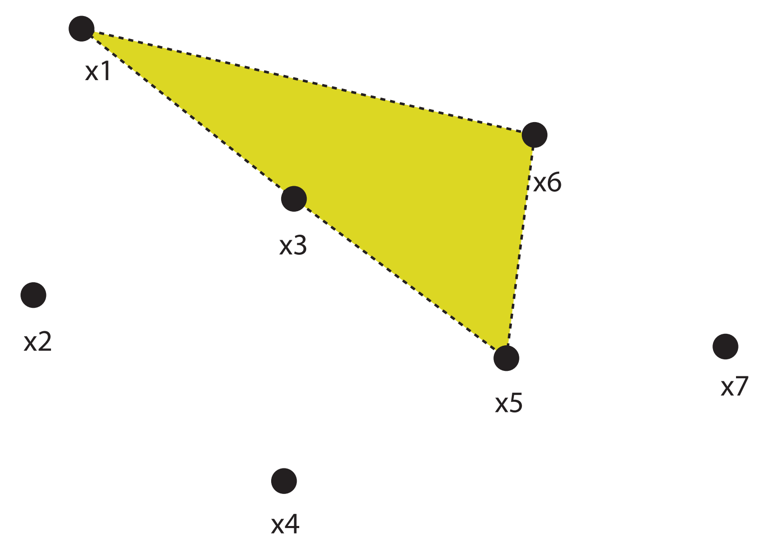
Example: Let $X = [x_1, x_{10}]$. Using the fact that the core is the intersection of the Pareto sets of all decisive coalitions, what is the core for the following rules:

$\frac{6}{10}$?

$\frac{8}{10}$?

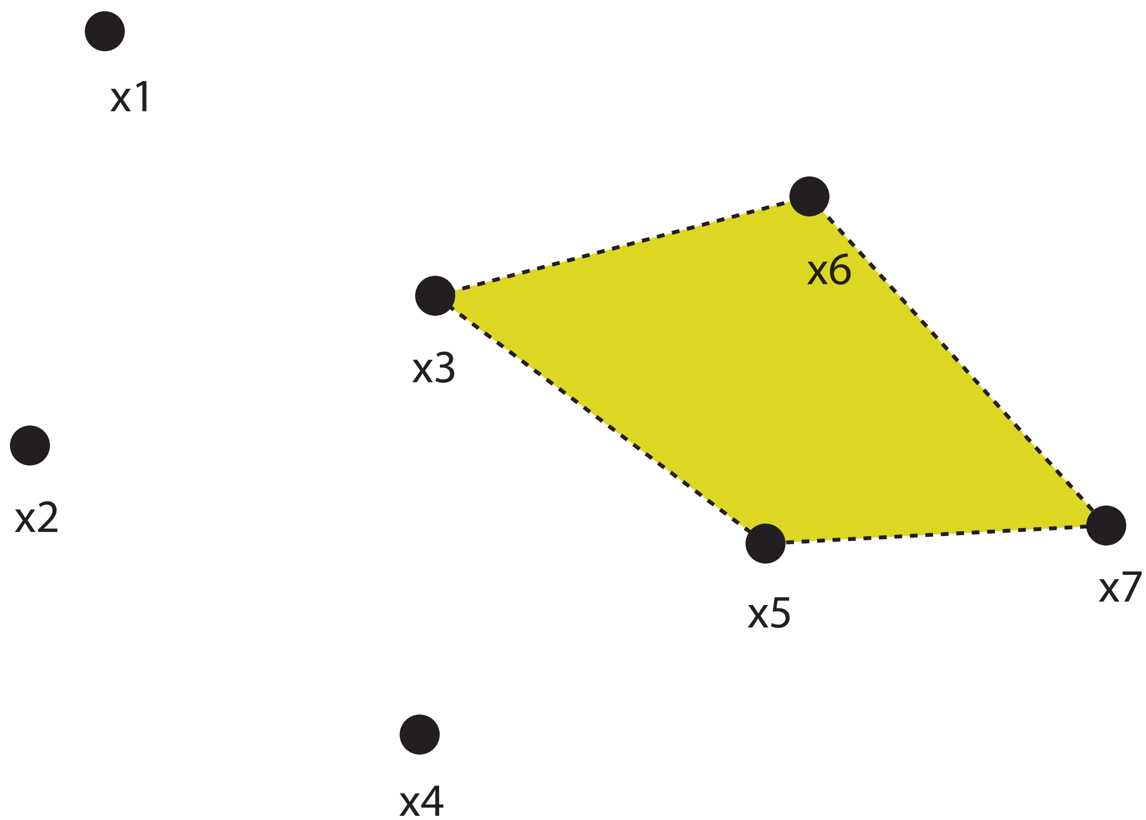
Unanimity rule?

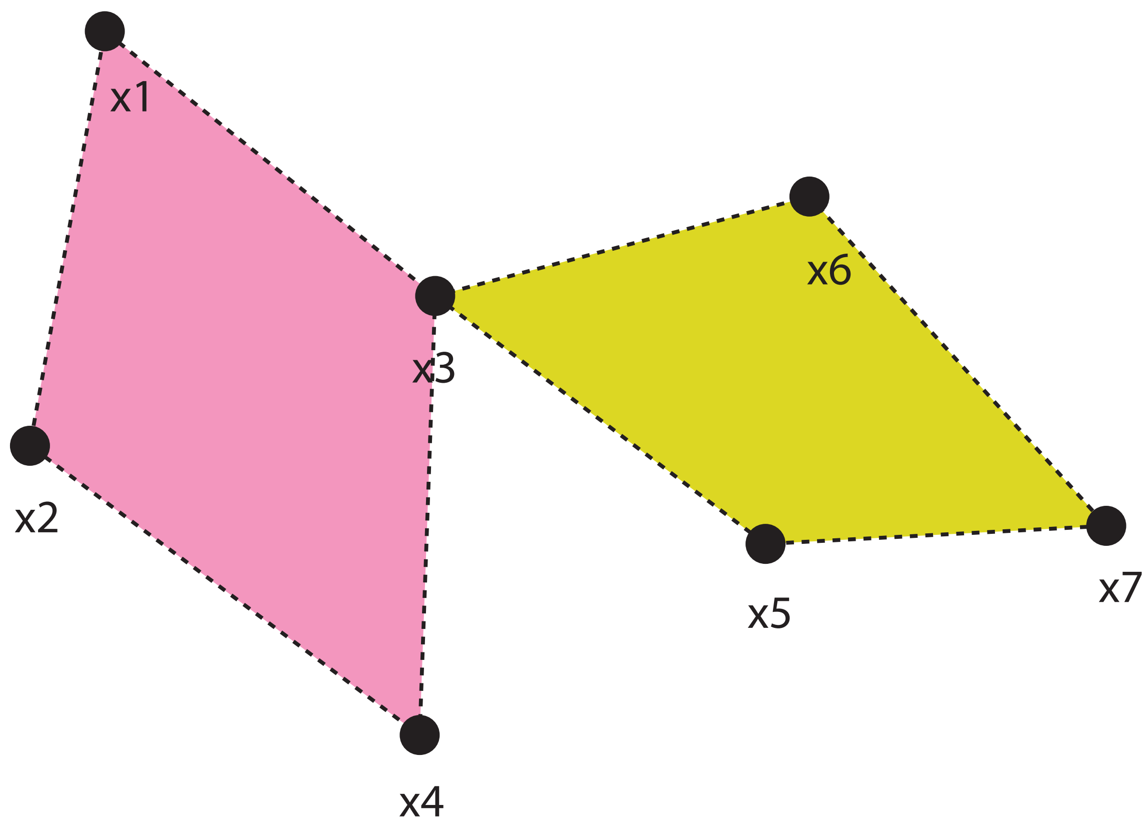
When people have Euclidean (circular) preferences, the Pareto set for any group of people is the convex hull of their ideal points (smallest convex set containing their ideal points).



Pareto set for $L = \{1, 3, 5, 6\}$ = convex hull of ideal points

By looking at the intersection of these Pareto sets, we can see that the majority rule core is empty in this example.


x1
x6
x3
x2
x7
x5
x4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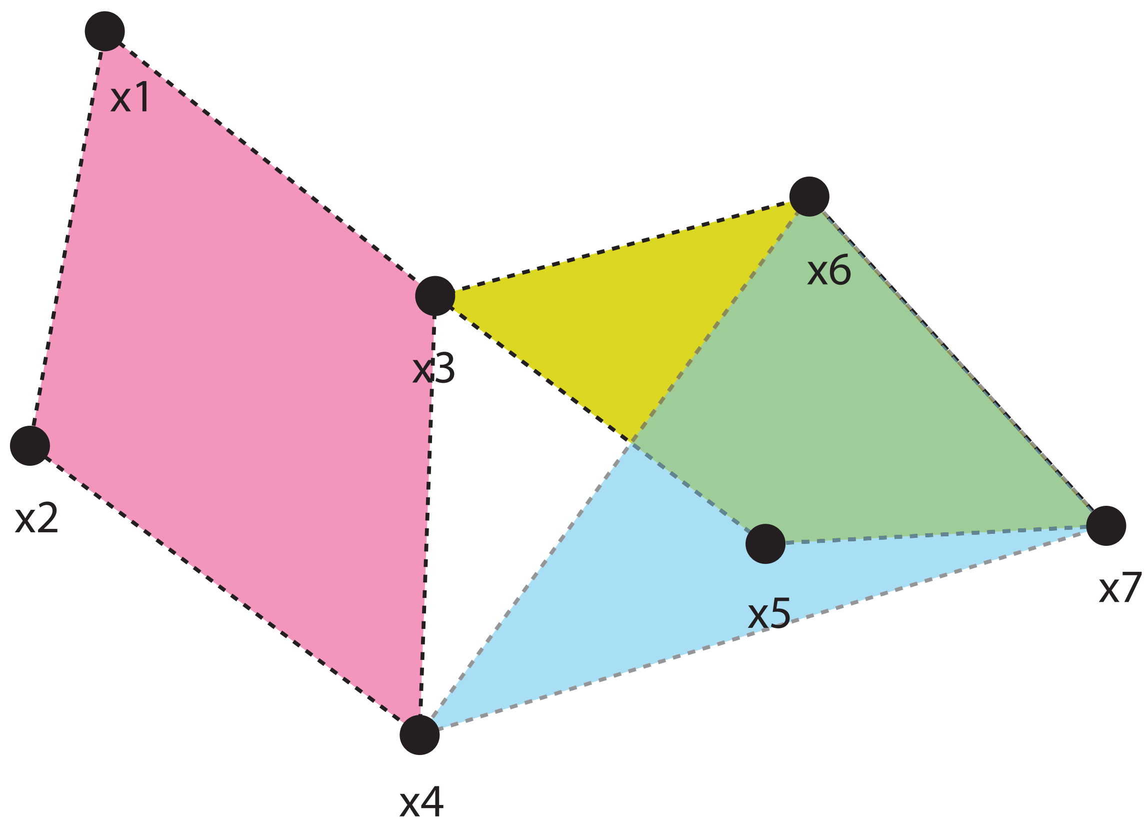
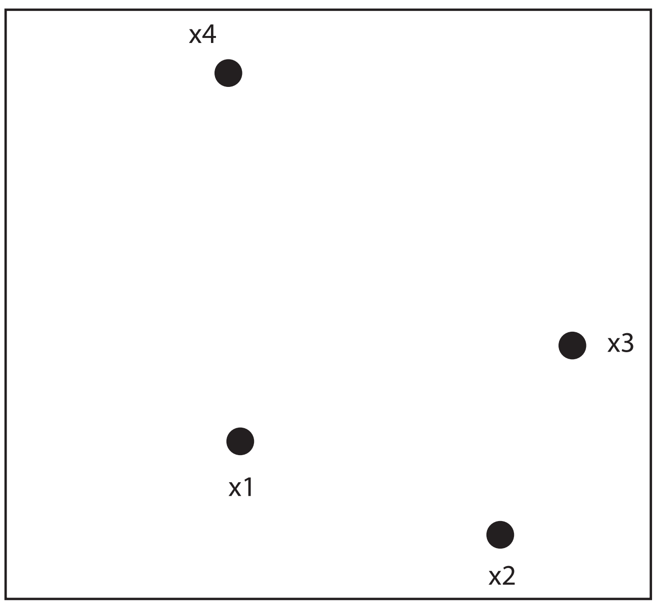
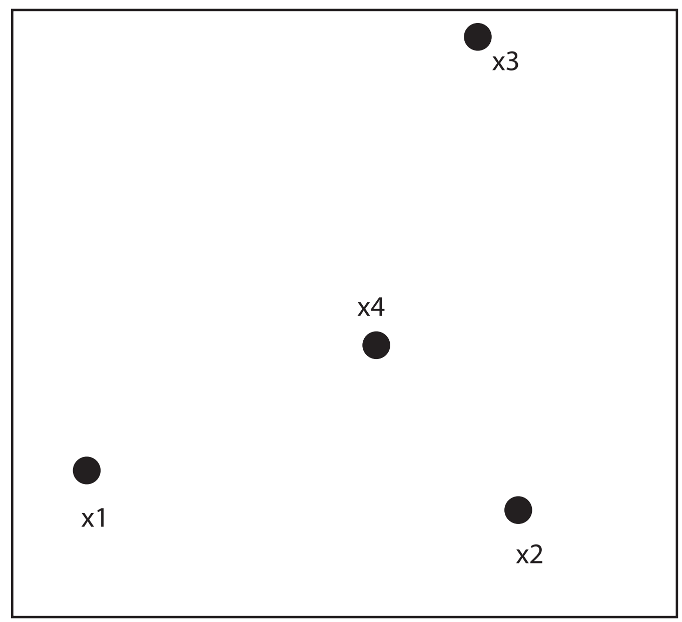

x1
x2
x3
x4
x5
x6
x7

Or result on the Nakamura number showed that if the dimensionality of the policy space less than or equal to $s(f) - 2$ implies the core is nonempty. Thus, in two dimensions, majority rule with 4 people has $s(f) = 4$, and thus, a non-empty core.

Find core for each distribution of ideal points, assuming circular prefs.



Example 1



Example 2

The Plott Conditions

Example: n=5, circular preferences: $u_i(x, y) = -(x_i - x)^2 - (y_i - y)^2$.