Fair Division in Theory and Practice

Ron Cytron (Computer Science)
Maggie Penn (Political Science)

Lecture 8: Power Indices

Weighted voting

In 1958, six W. Eurpoean countries formed the European Economic Community (EEC). Here was the vote breakdown:

Germany, France, Italy: 4 votes each
Belgium, Netherlands: 2 votes each
Luxembourg: 1 vote

The decision rule was 12 of 17 votes

Is there a dummy voter?
(A voter whose vote can never affect any outcome?)

Quantifying "power"

Whether or not a voter is a dummy is not monotone in the quota used or that voter's voting weight — it depends on the particular set of winning coalitions that a weighted voting system generates

Today we will discuss several quantitative measures of political power that take these configurations into account
(*power indices*)

Each measure tries to quantify a voter's control over outcomes

What does equalizing power mean?

Power indices can help us determine whether or not a voter's ability to affect outcomes corresponds in a logical way to his voting weight

How is "one person, one vote" best implemented in a weighted voting system? Equalizing vote share per person, or equalizing voting power per person?

Shapley-Shubik power index preliminaries (combinatorics)

Suppose we have n voters. How many different ways can we order them?

- $n = 2 : (12), (21)$, so 2 different ways
- $n = 3 : (123), (132), (213), (231), (312), (321)$, so 6
  - Look at $n = 2$ case; For each ordering there are three places we can place voter 3 (start, middle, end) — there are 3 times 2 possible orderings
- For $n = 4$, look at the $n = 3$ orderings; for each one, there are 4 different places we can place voter 4: $(\star 1 \star 2 \star 3 \star)$

Ordering voters

- Since there are 6 different orderings when $n = 3$, and 4 different places to put Voter 4 in each of those orderings, there are 4×6 orderings of 4 voters, or 24 orderings.
  - This is the same as $4 \times 3 \times 2 \times 1$
- When $n = 5$, there are 5 places to put Voter 5 in each of those orderings, so there are 5×24 orderings
  - This is $5 \times 4 \times 3 \times 2 \times 1$
- For n people, there $n \times (n-1) \times (n-2) \times ... \times 2 \times 1$ orderings of those people
- This is written $n!$

Pivotal players

Suppose we have 7 voters. There are 7! combinations of those voters. Take a particular one:

$$(3\ 5\ 1\ 6\ 7\ 4\ 2)$$

- One player will be **pivotal** for this particular ordering
- Picture a larger and larger coalition being formed as we move left to right
- Adding one person to the coalition we are forming will turn it from losing into a (minimal) winning coalition — that person is the pivotal voter for this ordering

Pivotal voters

(3 5 1 6 7 4 2)

Suppose each voter has one vote except for person 4, who has three votes

Suppose 5 votes are required for passage

Who is the pivotal voter for this ordering?

Who is pivotal for the ordering (3 4 1 6 7 5 2)?

Shapley-Shubik Indices

A person's Shapley-Shubik index is the fraction of orderings for which he is the pivotal voter

With n voters, Voter i's Shapley-Shubik index is:

$$\frac{\text{Number of orderings for which Voter } i \text{ is pivotal}}{n!}$$

Note that every person's index is between 0 and 1, and that the indices sum to 1

Question

Suppose there are three voters with the following voting weights: $(50, 49, 1)$, and that the voting quota is 51

What is each person's Shapley-Shubik index (SSI)?

Answer

Suppose there are three voters with the following voting weights: $(50, 49, 1)$, and that the voting quota is 51

What is each person's Shapley-Shubik index?

- (123): 2
- (132): 3
- (213): 1
- (231): 1
- (312): 1
- (321): 1

The voters' indices are $(\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$

What are the players' power indices when they have voting weights $(2, 2, 1)$ and 3 votes are required for passage?

A real-world example

European Economic Community 1958 (12 of 17 votes, $\approx 70\%$)

Germany, France, Italy: 4 votes each

Belgium, Netherlands: 2 votes each

Luxembourg: 1 vote

Country	Votes	Voting weight	SSI
Germany	4	.235	.233
France	4	.235	.233
Italy	4	.235	.233
Belgium	2	.12	.15
Netherlands	2	.12	.15
Luxembourg	1	.05	0

“Paradox of new members”

Suppose new members are added to the system and given votes, but that the total percentage of votes required for passage stays the same

We would expect the “power” of the original voters to become diluted or (stay the same) because their voting weights have decreased, but this is not the case

Revised EEC voting rule

1973 EEC (41 of 58 votes needed for passage, $\approx 70\%$)

Country	Votes	Voting weight	SSI
Germany	10	.17	.179
France	10	.17	.179
Italy	10	.17	.179
Belgium	5	.08	.081
Netherlands	5	.08	.081
Luxembourg	2	.03	.01
England	10	.17	.179
Denmark	3	.05	.057
Ireland	3	.05	.057

Luxembourg's power would have increased even if it had been left with just one vote

Banzhaf Indices

Introduced by attorney John Banzhaf in connection with a lawsuit involving Nassau County board of supervisors (1965)

We calculate this index by looking at all decisive coalitions. Look at each member of each coalition. If removing that member makes the coalition no longer decisive, then give him a point. That voter is a **critical voter** for that coalition.

When finished, divide by the total number of points, and you have calculated each person's Banzhaf power index

In-class question

Previous example of three voters with the following voting weights: $(50, 49, 1)$, and voting quota of 51

The decisive coalitions are:

$$(13), (12), (123)$$

In each coalition, who is a critical voter? (There may be several or none per coalition)

Answer

Previous example of three voters with the following voting weights: $(50, 49, 1)$, and voting quota of 51.

The decisive coalitions are:

$$(13), (12), (123)$$

- 1 and 3 are critical for (13)
- 1 and 2 are critical for (12)
- Only 1 is critical for (123)

The voters' Banzhaf indices are $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$

Recall that the voters' SSI's were $(\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$

In-class question

Consider a weighted "quota rule" (or q-rule), with quota $q = 6$ and $w_4 = 4, w_3 = 3, w_2 = 2$ and $w_1 = 1$. The winning coalitions are:

$$(4, 3), (4, 2), (4, 3, 2), (4, 3, 2, 1), (4, 3, 1), (4, 2, 1), (3, 2, 1)$$

Find the Banzhaf indices of the voters – remember, find the "critical voters" for each coalition, give them each a point, and sum up these points

Answer

Consider a weighted voting rule, with $q = 6$ and $w_4 = 4, w_3 = 3, w_2 = 2$ and $w_1 = 1$. The winning coalitions are:

$$(4,3),(4,2),(4,3,2),(4,3,2,1),(4,3,1),(4,2,1),(3,2,1)$$

The critical voters in each coalition are:

$$(4*,3*),(4*,2*),(4*,3,2),(4,3,2,1),(4*,3*,1),(4*,2*,1),(3*,2*,1*)$$

There are 12 critical votes, and the breakdown is:
4 (5), 3 (3), 2 (3), 1 (1), and so the Banzhaf indices are:

$5/12, 3/12, 3/12, 1/12.$

Thus, although 2 and 3 have different numbers of votes, they have the same voting power, given a (strong) assumption that any one of these situations in which a critical voter arises is equally likely (see Gelman and Katz, 2002)

Weighted voting in the U.S.

- Prior to 1960's several states had county-level Boards of Supervisors composed of elected supervisors from constituent cities or towns
  - These towns could differ greatly in population
- Following "one man, one vote" mandate (Reynolds v. Sims, 1964), most such states changed these legislative bodies
- 24 of NY's 57 counties did not, and instead opted for system of weighted voting

Arguments for weighted voting

- Allows existing, natural community to be the unit of representation, regardless of size
- Avoids problem of periodic apportionments / gerrymandering
- Enables existing officeholders (mayor, etc.) to double as the community's legislative representative

Argument against weights proportional to population

Banzhaf ("Weighted voting doesn't work," *Rutgers Law Review* 1965) challenged the constitutionality of weighted voting in Nassau County, NY; at the time there were 115 total votes & 58 were required for passage

Nassau county breakdown:
Hempstead 1: 31
Hempstead 2: 31
North Hempstead: 28
Oyster Bay: 21
Glen Cove: 2
Long Beach: 2

Are any members of the board dummy voters?

***Iannucci v. Board of Supervisors* (1967)**

NY's highest court ruled that any weighted voting system in New York had to be based on Banzhaf's method

- Implemented a computer-generated formula in Nassau and 23 other counties statewide that accounted for both population and the Banzhaf power of each board member
- Goal was to make population proportional to *voting power* as opposed to *voting weight*

A judicial decision based on findings of scholarly research

- A shift from (crude) focus on population equality to (sophisticated?) focus on legislator's ability to determine public policy

  "American courts have never clearly articulated exactly what it is they hope to equalize with [one man, one vote]... they have shown a zealous concern with something called 'equality' without coming to grips with the difficult notion of 'representation."

Grofman & Scarrow, "Weighted voting in New York," 1981

One problem with Banzhaf's method

It does not take into account the empirical likelihood of different coalitions forming

- Unit domination: a small number of units can join together to make everyone else dummies
- Modified schemes: very large cities could have multiple representatives share voting weight; under *Iannucci*, only individual member power was calculated
- After Iannucci, Hempstead had 70 votes total, divided 35-35 between two members; each member had 27.8% of the weight, so Hempstead overall had 55.6% power and 56% of the population
- If one member had 70 votes, they would have had 88.9% of the power, and the system would have been unconstitutional

Measurement problems

How should the match-up between population share and power share be measured?

- Traditionally in SMDs: (1) find ideal district size; (2) find difference between largest and smallest districts and this ideal; (3) take each difference and divide by size of ideal to express discrepancy as % of ideal; (4) find the range of the discrepancy
- Example: ideal=50, smallest = 45, largest =60
  - Smallest yields difference of 5 under, largest is 10 over
  - 5/50 (10%) too small and 10/50 (20%) too large
  - **Range of discrepancy is 30 percentage points**

Application to weighted voting & Banzhaf indices

- (1) ideal power for each legislator is pop%; (2) find largest and smallest discrepancies from ideals; (3) take each difference and divide by size of ideal to express discrepancy as % of ideal; (4) find the range of the discrepancy
- Example: Smallest power discrepancy= 18% (ideal 20%), largest power discrepancy=11% (ideal 10%)
  - Smallest yields difference of 2% under, largest is 1% over
  - 2/20 (10%) too small and 1/10 (10%) too large
  - **Range of discrepancy is 20 percentage points**

In *Franklin v Krause (1973)* the court mistakenly changed its precedent for how it determines a discrepancy

Smallest discrepancy: 18% actual (ideal 20%)
largest discrepancy: 11% actual (ideal 10%)

- Smallest yields difference of 2% under, largest is 1% over
- 2/20 (10%) too small and 1/10 (10%) too large
- **Range of discrepancy is 20 percentage points**
- **Method of measurement in *Franklin* would have found discrepancy of 3 percentage points**

In 1991 NYCLU filed suit that system violated one person, one vote

- 17 percent of residents were minorities and a third of registered voters were Democrats; almost every elected and appointed official in Nassau County was a white Republican

In a different 1989 case, NY Court ruled Banzhaf method failed because:

> "It does not attempt to inquire whether, in terms of how a legislature actually works in practice, the districts have equal power to affect a legislative outcome. This would be a difficult and ever-changing task, and its challenge is hardly met by a mathematical calculation that itself stops short of examining the actual day-to-day operations of the legislative body."

In 1991, Banzhaf argued Nassau misused his index. Since two members of six-person Board come from Hempstead and are Republicans who vote similarly on every issue, the application of the index confounds one of its main assumptions: that all winning coalitions are equally likely.

Voter power in large-scale elections

How should voter power be calculated in a presidential election, for example?

- In electoral college this is a two-step process
- First, every *state's* power index is calculated; given its share of the 538 total electoral college votes, how often can it cast a pivotal collection of votes.
- Next, each voter within that state's power index is calculated given the state population
- A voter's total power over the election is calculated (roughly) as the fraction of times his vote could be pivotal in the presidential election

Large states better off

Voters in large states are helped by the electoral college (in terms of their power indices), because state votes are cast as a block

- Suppose that electoral voting weights perfectly correspond to "power"
- Suppose a group of 3 voters had a voting weight of 3 (as a group). Another group of 1 voter has voting weight of 1.
- There are 8 combinations of yes-no votes among the three voters
- Assuming majority rule in the group of 3, each voter is critical in 50% of those cases
- Each voter is critical for 50% of the block of three votes (1.5); The one voter with one vote is critical for 1 vote

APPENDIX*

TABLE I

PRESENT ELECTORAL COLLEGE

State Name (1)	*Population 1960 Census*	*Electoral Vote 1964*	*Relative Voting Power (2)*	*Percent Excess Voting Power (3)*	*Percent Deviation From Average Voting Power (4)*
Alabama	3266740.	10	1.632	63.2	-3.0
Alaska	226167.	3	1.838	83.8	9.2
Arizona	1302161.	5	1.281	28.1	-23.9
Arkansas	1786272.	6	1.315	31.5	-21.9
California	15717204.	40	3.162	216.2	87.9
Colorado	1753947.	6	1.327	32.7	-21.1
Connecticut	2535234.	8	1.477	47.7	-12.2
Delaware	446292.	3	1.308	30.8	-22.3
Dist. of Columbia	763956.	3	1.000	.0	-40.6
Florida	4951560.	14	1.870	87.0	11.1
Georgia	3943116.	12	1.789	78.9	6.3
Hawaii	632772.	4	1.468	46.8	-12.8
Idaho	667191.	4	1.429	42.9	-15.1
Illinois	10081158.	26	2.491	149.1	48.0
Indiana	4662498.	13	1.786	78.6	6.1
Iowa	2757537.	9	1.596	59.6	-5.2
Kansas	2178611.	7	1.392	39.2	-17.3
Kentucky	3038156.	9	1.521	52.1	-9.6
Louisiana	3257022.	10	1.635	63.5	-2.9
Maine	969265.	4	1.186	18.6	-29.5
Maryland	3100689.	10	1.675	67.5	-.4
Massachusetts	5148578.	14	1.834	83.4	9.0
Michigan	7823194.	21	2.262	126.2	34.4
Minnesota	3413864.	10	1.597	59.7	-5.1
Mississippi	2178141.	7	1.392	39.2	-17.3
Missouri	4319813.	12	1.710	71.0	1.6
Montana	674767.	4	1.421	42.1	-15.5
Nebraska	1411330.	5	1.231	23.1	-26.9
Nevada	285278.	3	1.636	63.6	-2.8
New Hampshire	606921.	4	1.499	49.9	-10.9
New Jersey	6066782.	17	2.063	106.3	22.6
New Mexico	951023.	4	1.197	19.7	-28.9
New York	16782304.	43	3.312	231.2	96.8
North Carolina	4556155.	13	1.807	80.7	7.4
North Dakota	632446.	4	1.468	46.8	-12.8
Ohio	9706397.	26	2.539	153.9	50.9
Oklahoma	2328284.	8	1.541	54.1	-8.4
Oregon	1768687.	6	1.321	32.1	-21.5
Pennsylvania	11319366.	29	2.638	163.8	56.8
Rhode Island	859488.	4	1.259	25.9	-25.2
South Carolina	2382594.	8	1.524	52.4	-9.5
South Dakota	680514.	4	1.415	41.5	-15.9
Tennessee	3567089.	11	1.721	72.1	2.3
Texas	9579677.	25	2.452	145.2	45.7
Utah	890627.	4	1.237	23.7	-26.5
Vermont	389881.	3	1.400	40.0	-16.8
Virginia	3966949.	12	1.784	78.4	6.0
Washington	2853214.	9	1.569	56.9	-6.8
West Virginia	1860421.	7	1.506	50.6	-10.5
Wisconsin	3951777.	12	1.788	78.8	6.2
Wyoming	330066.	3	1.521	52.1	-9.6

(1) Includes the District of Columbia.
(2) Ratio of voting power of citizens of state compared with voters of the most deprived state.
(3) Percent by which voting power exceeds that of the most deprived voters (deviations).
(4) Percent by which voting power deviated from the average of the figures in column 4. Minus signs in Tables I-III indicate less than average voting power.

* STUDY PREPARED BY JOHN F. BANZHAF III, 100 PARK AVE., N.Y.C., BASED UPON WORK REPORTED IN: 19 RUTGERS LAW REVIEW 317-43 (1965) & 75 YALE LAW JOURNAL 1309-38 (1966) // COMPUTER CALCULATIONS MADE UNDER DIRECTION OF AUTHOR BY PROF. MARTIN A. JACOBS, DEPT. OF MATH., FAIRLEIGH DICKINSON UNIV., TEANECK, N.J., ON UNIVERSITY'S G.E.-235 TIME-SHARED COMPUTER SYSTEM.

The "Banzhaf Fallacy," Howard Margolis

Large states will necessarily look as though they have more "power" than small states when we take the likelihood of a vote to be a coin flip

In reality, the most powerful states theoretically are generally not swing states (which are the most powerful in practice)

A large literature has looked at power indices empirically, to calculate a voter or state's likelihood of being pivotal *conditional on the empirical likelihood of various coalitions forming*

One last example

Consider weighted voting with weights $(5, 3, 1, 1, 1)$ and quota 8. The Banzhaf indices of the voters are

$$(\frac{9}{19}, \frac{7}{19}, \frac{1}{19}, \frac{1}{19}, \frac{1}{19})$$

If Voter 1 gives one of his votes to Voter 2, to yield the new set of weights $(4, 4, 1, 1, 1)$ what are the new Banzhaf indices for the voters?