Fair Divsion in Theory and Practice

Ron Cytron (Computer Science)
Maggie Penn (Political Science)

Lecture 6: Last Diminsher and a nearly envy-free algorithm

Review: Dubins–Spanier Moving Knife Algorithm, arbitrary n

Procedure:

- A knife is moved slowly across the cake, from left to right.
- Any player without an assigned piece can say "stop".
- That player is assigned the piece to the left of the knife.
- When only one player remains, that player is assigned the piece to the right of the knife (π_R).

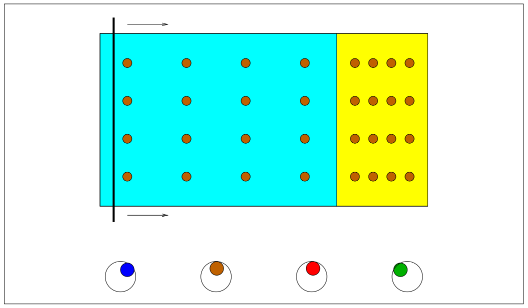
Strategy: If π_L is the cake currently to the left of the moving knife, then any player P_i must say stop when

$$v_i(\pi_L) = \frac{1}{n}$$

to guarantee P_i a proportional piece of the cake.

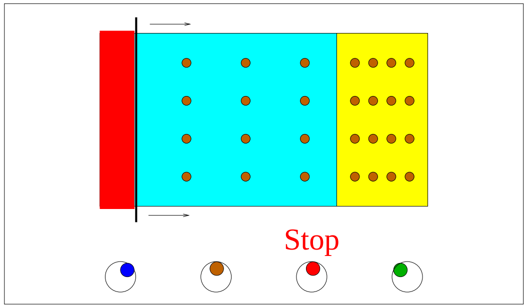
Moving Knife Algorithm, arbitrary n

The knife starts to move across the cake, from left to right. There are four players in this example, distinguished by the color of their eyeballs.

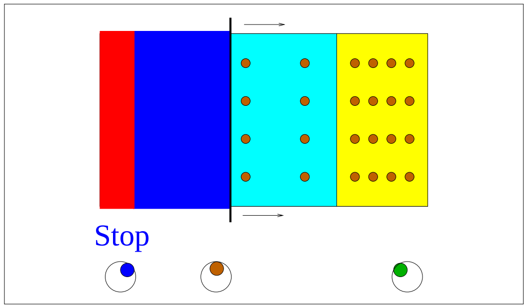


Suppose the red player is the first to say "stop". That player is assigned the piece of cake to the left of the knife, also shown in red. Because red called stop first:

$$\forall\ i \neq \text{red}\ v_i(\text{red piece}) < \frac{1}{4}$$

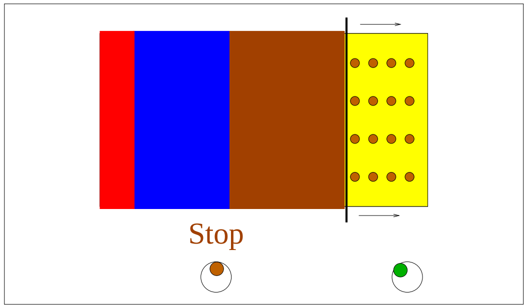


The red player drops out of the algorithm. Suppose blue is the next to say "stop". The blue piece is assigned as shown.

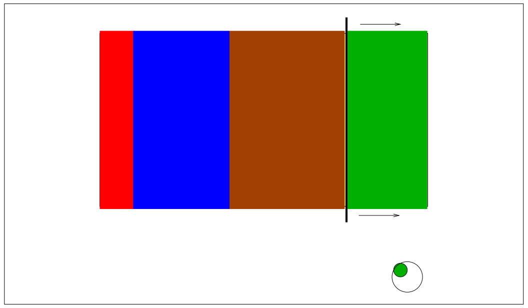


Moving Knife Algorithm, arbitrary n

Two players remain, brown and green. If brown calls stop first, then the brown piece is assigned as shown.



The green player never called stop, so as the last player, the green player receives the piece to the right of the stopped knife.



Moving Knife Algorithm, arbitrary n

- Is there always enough cake for everybody?
  - When done, is the following true?

$$\forall\ i\ v_i(\pi_i) \geq \frac{1}{n}$$

- What happens if some P_i tries to be greedy and does not say stop the first time $v_i(\pi_L) = \frac{1}{n}$?
- Can envy develop? If so can it exist between any players, or only between some?
  - Hint: Can the player who receives π_R envy anybody?
- Is the result necessarily equitable?
- Is the result necessarily stable?

Today, two things

- Move cake instead of knives
- A nearly envy-free version of moving knife

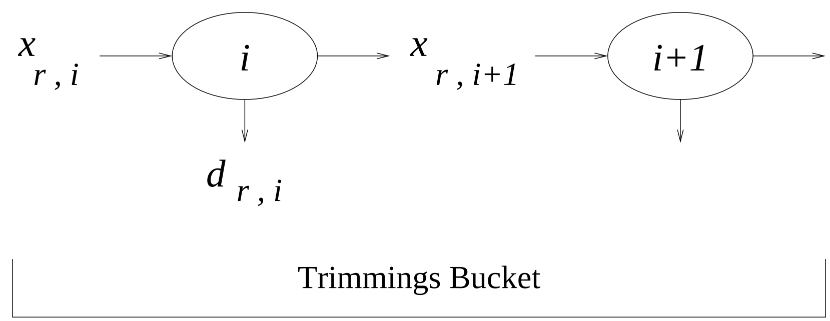
Banach–Knaster Last Diminisher

- Move the cake instead of the knife
- Akin to Divide and Choose
- Not a "continuous" algorithm
  - But a person cuts the cake at a point related to his or her preferences
- Works in *rounds*:
  - Call these rounds $R_1, R_2, \ldots, R_n$
  - In each round, all remaining players consider the cake as it goes by
  - At the end of each round, one person receives cake and is no longer allowed to cut any cake

Banach–Knaster Last Diminisher

In R_r:

- P_i receieves a piece of cake $x_{r,i}$
- P_i creates piece $x_{r,i+1}$ by *trimming* away $d_{r,i}$, $v_i(d_{r,i}) \geq 0$
  - If $v_i(d_{r,i}) = 0$, no diminishment
  - Otherwise, diminishment

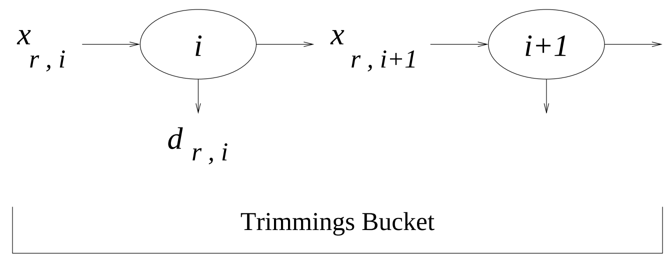


- The last person P_p to diminish the cake receives the piece $x_{r,p+1}$
- The trimmings are collected in a bucket, and then concatentated to make the piece $x_{r+1,1}$ for the next round.
- The algorithm runs for $n - 2$ rounds. The remaining two players then use Divide and Choose.

Banach–Knaster Last Diminisher

Strategy: P_i must trim the cake to pass on a proportinal piece.

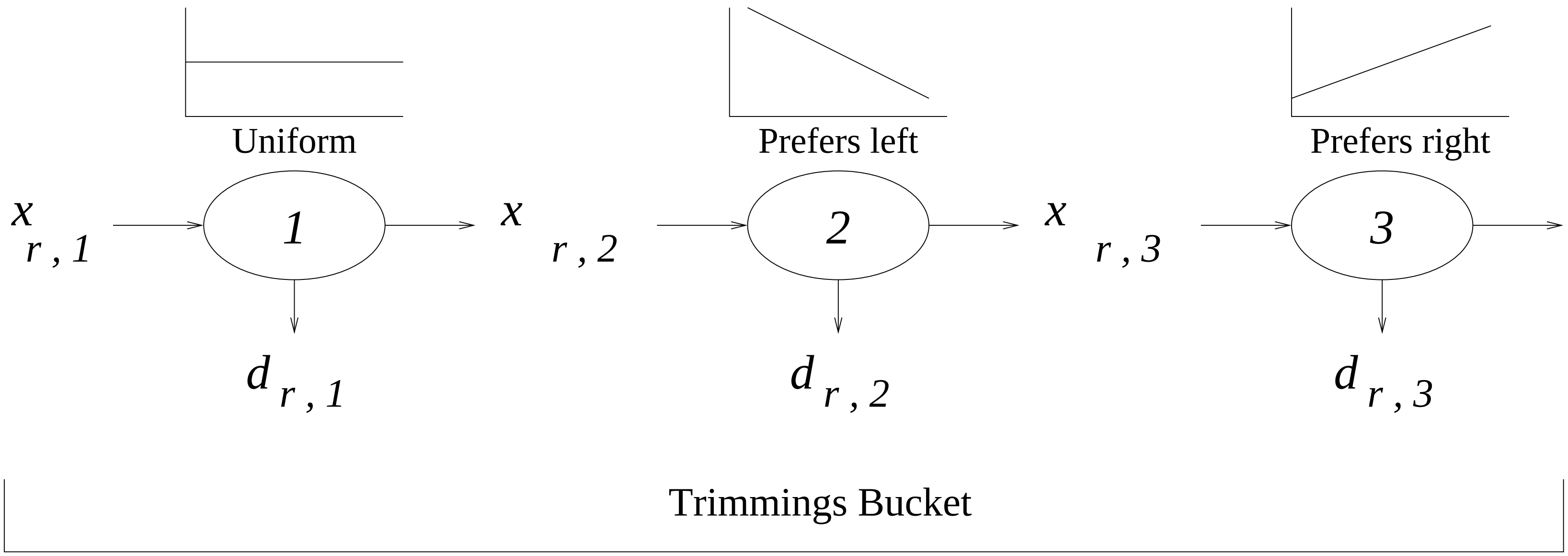
- If $v_i(x_{r,i}) > \frac{1}{n}$ then P_i must trim the cake such that $v_i(x_{r,i+1}) = \frac{1}{n}$
- Otherwise, don't trim: $v_i(x_{r,i+1}) = v_i(x_{r,i})$



Why?

- Trim too much, and P_i may receive a piece of cake whose value is less than proportional.
- Trim too little, or fail to trim at all, and P_i may enable another player to receive so much cake that a proportional piece will not be available in the future for P_i.

Example for Last Diminisher

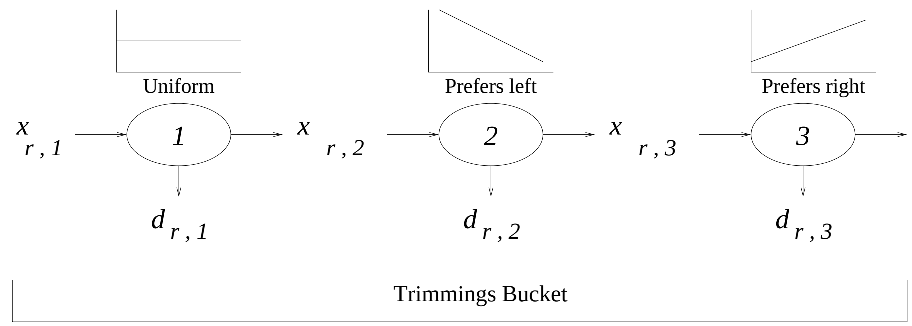


Consider three players as shown above. P_1 views the cake uniformly, P_2 prefers the left side of the cake, and P_3 prefers the right side of the cake.

Last Diminisher Round R_1

P_1 receives the entire cake and is forced to trim:

$$v_1(x_{1,1}) = 1$$
$$v_1(x_{1,2}) = \frac{1}{3}$$
$$v_1(d_{1,1}) = \frac{2}{3}$$



P_2 receives a larger than proprtional piece so must also trim:

$$v_2(x_{1,2}) = \frac{1}{2}\text{(say)}$$
$$v_2(x_{1,3}) = \frac{1}{3}$$
$$v_2(d_{1,2}) = \frac{1}{6}$$

P_3 receives $x_{1,3}$ which is, say, too small, so P_3 passes the cake on. The last diminisher was P_2 who then receives $x_{1,3}$.

Last Diminisher: Two players remain

Players P_1 and P_3 now apply DIVIDE AND CHOOSE to the cake in the trimming bucket.

Claim: Both remaining players see the remaining cake as having value at least $\frac{2}{3}$.

- P_1 sees $\frac{2}{3}$ in the bucket because of trimming $d_{1,1}$. The addition of $d_{1,2}$ could increase P_1's view of the remaining cake.
- P_3 passed on $x_{1,3}$, so its value must have been less than $\frac{1}{3}$. Thus, the reamining cake must have at least value $\frac{2}{3}$

Thus, by DIVIDE AND CHOOSE each player can get at least $\frac{1}{2}$ of a value seen by each to be at least $\frac{2}{3}$. Thus, each receives a proportional amount of cake.

Last Diminisher: for you to prove

- Anybody receiving a piece of cake views the value of that cake as being at least $\frac{1}{n}$.
- In each round, somebody will trim the cake.
- When two players remain, each will see the remaining cake as having value of at least $\frac{2}{n}$.

From proportional to (nearly) envy free

- There are simple envy-free cake cutting algorithms for up to 4 players.
- Brams claims an envy-free algorithm for aribitrary n. It is not simple.
- However, we can adapt Dubins–Spanier to obtain an *nearly* envy-free algorithm.

Dubins–Spanier, nearly envy free

- Run DUBINS–SPANIER as usual, *except* that a player P_i can call "stop" again if he or she sees a piece whose value is ϵ better than π_i. P_i then trades in his or her current piece for the piece to the left of the knife.
- As a rule, a player can say "stop" at most $\frac{1}{\epsilon}$ times.

Strategy: Player P_i calls "stop" when seeing value $\frac{1}{n}$ or a value that is ϵ bigger than P_i's current piece.

Almost envy free, for you to prove

Theorem 1 *Using the stated strategy, each player ends up with a piece that is at most ϵ smaller than the largest piece they see among all of the players.*

Proof: Homework ■

In other words, this algorithm is ϵ-away from being envy free!

- Can this be applied to LAST DIMINISHER?
- Why the limiting rule of $\frac{1}{\epsilon}$ stops per player?