# Fair Division in Theory and Practice

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*Lecture 4: The List Systems of Proportional Representation*

## Saari's "milk, wine, beer" example

Thirteen people with the following preferences must decide on one choice of beverage for the group:

| Number of voters | Preference ranking |
|---|---|
| 4 | M W B |
| 2 | W B M |
| 1 | B M W |
| 2 | M B W |
| 4 | B W M |

| Number of voters | Preference ranking |
|---|---|
| 4 | M W B |
| 2 | W B M |
| 1 | B M W |
| 2 | M B W |
| 4 | B W M |

What is the outcome under plurality rule?

Under a “vote for two” rule?

Under Borda Count?

| Number of voters | Preference ranking |
|---|---|
| 4 | M W B |
| 2 | W B M |
| 1 | B M W |
| 2 | M B W |
| 4 | B W M |

What is the outcome under plurality rule? **MBW by 6:5:2**

Under a “vote for two” rule? **WBM by 10:9:7**

Under Borda Count? **BMW by 14:13:12**

The Borda count, plurality rule and the "vote for two" rule
(a variant of the block vote) are all real-world voting rules.

Is the selection of a winner more a function of the voting procedure used than of the preferences of the voters?

What does it mean to "represent the preferences of the voters"?

How can we evaluate these systems, given that they all appear reasonable?

## The pairwise vote (the Condorcet method): a bad decision procedure?

| Voter | Preference ranking |
|---|---|
| 1 | A$\succ$B$\succ$C |
| 2 | B$\succ$C$\succ$A |
| 3 | C$\succ$A$\succ$B |

What is the majority's choice between A and B?

Between B and C?

Between C and A?

| Voter | Preference ranking |
|---|---|
| 1 | A≻B≻C |
| 2 | B≻C≻A |
| 3 | C≻A≻B |

A simple majority vote between pairs of alternatives yields a binary relation that describes the "preferences of the majority":

$$A \succ B \qquad B \succ C \qquad C \succ A$$

This is called **Condorcet's Paradox**, and implies that the "will of the majority" will not necessarily yield a "best" outcome. Here, majority preferences are described by a binary relation that is not transitive.

## Condorcet's paradox can be generalized (number of people, number of alternatives)

To illustrate, suppose that there are $n$ voters and we would like a decision procedure to satisfy a property called *minimal democracy*:

If for some alternative $x$ there is a different $y$ such that $n-1$ people strictly prefer $y$ to $x$, then $x$ can't be the social choice.

With $n$ people and at least $n$ alternatives to choose from, no procedure can be *guaranteed* to satisfy minimal democracy.

$$\text{Person } 1: \quad x_1 \succ x_2 \succ x_3 \succ ... \succ x_{n-1} \succ x_n$$

$$\text{Person } 2: \quad x_2 \succ x_3 \succ x_4 \succ ... \succ x_n \succ x_1$$

$$\text{Person } 3: \quad x_3 \succ x_4 \succ x_5 \succ ... \succ x_1 \succ x_2$$

...

$$\text{Person } n: x_n \succ x_1 \succ x_2 \succ ... \succ x_{n-2} \succ x_{n-1}$$

This generates a majority preference cycle $x_1 \succ x_2 \succ ... \succ x_n \succ x_1$ in which $n - 1$ people share the same preferences over every pairwise vote (for any alternative chosen, $n - 1$ people agree to prefer something else)

## Is the Condorcet method of taking pairwise votes to determine a social outcome a *good* procedure?

The 1998 Minnesota gubernatorial election

| | |
|---|---|
| Dem. Hubert Humphrey (Atty General) | 28% |
| Rep. Norm Coleman (St. Paul mayor) | 35% |
| Ref. Jesse "The Body" Ventura | 37% |

| | | | | | |
|---|---|---|---|---|---|
| 28% | Humphrey | $\succ$ | Coleman | $\succ$ | Ventura |
| 35% | Coleman | $\succ$ | Humphrey | $\succ$ | Ventura |
| 37% | Ventura | $\succ$ | Coleman | $\succ$ | Humphrey |

Coleman is a *Condorcet winner*, a candidate that defeats every other candidate in a pairwise vote.

Ventura is a *Condorcet loser*, a candidate that loses to every other candidate in a pairwise vote

## ***Condorcet consistency*: the ability to select a Condorcet winner if one exists**

The Minnesota election used plurality rule: each alternative got one "point" for each ballot it was at the top of.

Condorcet's method ranks the alternatives: $C \succ H \succ V$, while plurality rule ranks the alternatives $V \succ C \succ H$.

Plurality rule is not Condorcet consistent; not only did it not select a Condorcet winner, but it selected a Condorcet loser.

# What do these examples tell us?

- They appear to imply that anything is possible, given the appropriate choice of electoral system.
- Electoral system choice is meaningful because of the fact that no procedure is perfect (and because reasonable systems can produce very different outcomes)
- Electoral system designers must define the criteria that they want their system to satisfy

## Classifying electoral systems

1. District magnitude
   - The number of seats to be filled by the particular election, or "number of winners"
2. Ballot structure
   - Ordinal or categorical; whether voters rank candidates or check boxes
3. Electoral formula
   - The mathematical formula that translates ballots into seats; a coarse categorization would be *plurality*, *proportional*, and *majority*.

## Other ways to classify electoral systems would focus on their outputs

Whether the systems produce proportional results

- Ways of distorting the proportionality of a system include *malapportionment*, *electoral formula*, and *electoral thresholds*.

Whether the systems induce strategic behavior (plurality systems are notorious in this – the Nader effect)

Whether there are many invalid ballots (i.e. whether the system is simple enough for people to understand and use)

## The List Systems of Proportional Representation

Some number of seats are to be distributed to parties that run in an election and voters vote for a party

Goal is to allocate seats "proportionally" to the parties based on their shares of the total votes, so that for each party:

$$\% \text{ of total votes cast for party} = \% \text{ of total seats given to party}$$

## How they work

- For each constituency, each party draws up a list of candidates; length of list depends on DM
- Most countries draw up sub-national constituencies for each election; some countries hold nationwide election (Israel, Iraq's first election post-Hussein)
- Depending on % of vote each party receives and formula used, seats are distributed to the parties
- DM $> 1$ (always), categorical ballot (though there is a lot of variation)

## Two types of formulas for granting seats to parties

Largest remainder systems

- Seat allocation occurs in two rounds
- Formula uses subtraction (Hamilton's method)

Highest average systems

- Seat allocation occurs over many rounds
- Formula uses division (Divisor methods)

## Largest remainder systems

- All utilize an *electoral quota* (# of votes that translate into a seat)
- Seat allocation occurs in two rounds
- In first round, parties meeting the quota are awarded seats
- The (quota) * (# seats awarded) is subtracted from each party's total vote
- In second round parties left with the greatest number of votes are awarded any leftover seats

## About Quotas

First, the total valid vote is calculated. The formulas are then:

- Hare quota (Hamilton's Method): $\frac{\text{votes}}{\text{seats}}$
- Droop quota: $\frac{\text{votes}}{\text{seats}+1} + 1$
- Imperiali quota: $\frac{\text{votes}}{\text{seats}+2}$

Drop remainder for Droop quota, not for others.

Note: Hare > Droop > Imperiali, and
*bigger quotas benefit smaller parties*

**Hare Example: 100 votes, 5 seats $\Rightarrow$ Quota = ?**

| Party | Round 1 | Seats won | Round 2 | Seats won | Total |
|---|---|---|---|---|---|
| A | 36 | | | | |
| B | 31 | | | | |
| C | 15 | | | | |
| D | 12 | | | | |
| E | 6 | | | | |

**Imperiali Example: 100 votes, 5 seats $\Rightarrow$ Quota = ?**

| Party | Round 1 | Seats won | Round 2 | Seats won | Total |
|---|---|---|---|---|---|
| A | 36 | | | | |
| B | 31 | | | | |
| C | 15 | | | | |
| D | 12 | | | | |
| E | 6 | | | | |

**Hare Example: 100 votes, 5 seats ⇒ Quota = 20**

| Party | Round 1 | Seats won | Round 2 | Seats won | Total |
|---|---|---|---|---|---|
| A | 36 | | | | |
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**Hare Example: 100 votes, 5 seats ⇒ Quota = 20**

| Party | Round 1 | Seats won | Round 2 | Seats won | Total |
|---|---|---|---|---|---|
| A | 36 | 1 | | | |
| B | 31 | 1 | | | |
| C | 15 | 0 | | | |
| D | 12 | 0 | | | |
| E | 6 | 0 | | | |

**Imperiali Example: 100 votes, 5 seats ⇒ Quota = ?**

| Party | Round 1 | Seats won | Round 2 | Seats won | Total |
|---|---|---|---|---|---|
| A | 36 | | | | |
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**Hare Example: 100 votes, 5 seats $\Rightarrow$ Quota = 20**

| Party | Round 1 | Seats won | Round 2 | Seats won | Total |
|---|---|---|---|---|---|
| A | 36 | 1 | 16 | | |
| B | 31 | 1 | 11 | | |
| C | 15 | 0 | 15 | | |
| D | 12 | 0 | 12 | | |
| E | 6 | 0 | 6 | | |

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**Hare Example: 100 votes, 5 seats ⇒ Quota = 20**

| Party | Round 1 | Seats won | Round 2 | Seats won | Total |
|---|---|---|---|---|---|
| A | 36 | 1 | 16 | 1 | 2 |
| B | 31 | 1 | 11 | 0 | 1 |
| C | 15 | 0 | 15 | 1 | 1 |
| D | 12 | 0 | 12 | 1 | 1 |
| E | 6 | 0 | 6 | 0 | 0 |

**Imperiali Example: 100 votes, 5 seats ⇒ Quota = ?**

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|---|---|---|---|---|---|
| A | 36 | 1 | 16 | 1 | 2 |
| B | 31 | 1 | 11 | 0 | 1 |
| C | 15 | 0 | 15 | 1 | 1 |
| D | 12 | 0 | 12 | 1 | 1 |
| E | 6 | 0 | 6 | 0 | 0 |

**Imperiali Example: 100 votes, 5 seats ⇒ Quota = 14.2**

| Party | Round 1 | Seats won | Round 2 | Seats won | Total |
|---|---|---|---|---|---|
| A | 36 | 2 | $\approx 8$ | 0 | 2 |
| B | 31 | 2 | $\approx 3$ | 0 | 2 |
| C | 15 | 1 | $\approx 1$ | 0 | 1 |
| D | 12 | 0 | 12 | 0 | 0 |
| E | 6 | 0 | 6 | 0 | 0 |

# Highest Average systems

Parties' vote totals are divided by a series of divisors to form a table of averages

If there are $k$ seats to be allocated, the seats are assigned to the parties with the $k$ highest numbers in the table

There are two commonly used sequences of numbers (Sainte Laguë is rare)

- D'Hondt (Jefferson's Method): 1, 2, 3, 4, 5, 6, 7, ...
- Modified Sainte Laguë: 1.4, 3, 5, 7, 9, ...
- Sainte Laguë (Webster's Method): 1, 3, 5, 7, 9, ...

**D' Hondt Example: 100 votes, 5 seats**

| Party | Votes/1 | Votes/2 | Votes/3 | Total |
|---|---|---|---|---|
| A | 36 | 18 | 12 | |
| B | 31 | 15.5 | 10.3 | |
| C | 15 | 7.5 | 5 | |
| D | 12 | 6 | 4 | |
| E | 6 | 3 | 2 | |

**Sainte Laguë Example: 100 votes, 5 seats**

| Party | Votes/1 | Votes/3 | Votes/5 | Total |
|---|---|---|---|---|
| A | 36 | 12 | 7.2 | |
| B | 31 | 10.3 | 6.2 | |
| C | 15 | 5 | 3 | |
| D | 12 | 4 | 2.4 | |
| E | 6 | 2 | 1.2 | |

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## D' Hondt Example: 100 votes, 5 seats

| Party | Votes/1 | Votes/2 | Votes/3 | Total |
|---|---|---|---|---|
| A | 36 | 18 | 12 | 2 |
| B | 31 | 15.5 | 10.3 | 2 |
| C | 15 | 7.5 | 5 | 1 |
| D | 12 | 6 | 4 | 0 |
| E | 6 | 3 | 2 | 0 |

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| A | 36 | 12 | 7.2 | 2 |
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| C | 15 | 5 | 3 | 1 |
| D | 12 | 4 | 2.4 | 1 |
| E | 6 | 2 | 1.2 | 0 |

## What are these different systems accomplishing?

- Different PR formulas minimize different standards of proportionality
- D'Hondt (Jefferson) minimizes "over-representation of the most represented party"
- What does this mean?

| Party | Votes | Votes/2 | Votes/3 | Votes/4 | seats |
|---|---|---|---|---|---|
| A | 60000 | 30000 | 20000 | 15000 | 4 |
| B | 28000 | 14000 | | | 1 |
| C | 12000 | | | | 0 |
| Total | 100000 | | | | 5 |

| Party | Votes | Votes/2 | Votes/3 | Votes/4 | seats |
|---|---|---|---|---|---|
| A | 60000 | 30000 | 20000 | 15000 | 4 |
| B | 28000 | 14000 | | | 1 |
| C | 12000 | | | | 0 |
| Total | 100000 | | | | 5 |

- A has 60% of the vote, why not 3 seats?
- A's "overrepresentation" is 80% seats to 60% votes; $\frac{80}{60} \approx 1.33$
- If B got another seat it would be overrepresented by $\frac{40}{28} \approx 1.43$
- If C got another seat it would be overrepresented by $\frac{20}{12} \approx 1.67$

D'Hondt chooses an allocation that minimizes this term

# Indices of proportionality

- An index assigns a number to the outcome of an electoral system (vote/seat shares) that says something about how (dis)proportional the system is
- If DM=1 or DM=$\infty$ all formulas will yield the same outcome; small to intermediate DMs matter
- Different notions of proportionality will yield different indices; many formulas will conceptually characterize an index, and will minimize it
- D'Hondt minimizes an index that is "seat share-to-vote share ratio of the most over-represented party"

## Which index to choose?

- All indices use data consisting of $i = 1, ..., n$ total parties
- The *vote shares* $v_i$ of those parties
  (votes for party / total votes cast)
- And the *seat shares* $s_i$ of the parties
  (seats party wins / total seats allocated)

Given $v = (v_1, ..., v_n)$ and $s = (s_1, ..., s_n)$ how would you construct an index of proportionality?

## More indices

- Loosemore-Hanby Index
- Given a collection of *vote shares* $v_i$ and *seat shares* $s_i$ for parties $i = 1, ..., n$

$$\frac{\sum_i |v_i - s_i|}{2}$$

- This is always minimized by largest remainder with Hare quota (a.k.a. Hamilton's method)
- It's subject to all of the paradoxes that Hamilton's method is subject to (Alabama, population, new states, etc.)

## Criticisms of Loosemore-Hansby

$$\frac{\sum_i |v_i - s_i|}{2}$$

- Can you see any downsides to using an index like this (in addition to the fact that it suffers from paradoxes)?

## Criticisms of Loosemore-Hansby

$$\frac{\sum_i |v_i - s_i|}{2}$$

- It can't discriminate between *a lot* of parties having *small* vote-seat share differences and *a few* parties having *big* vote-seat differences
- How would you deal with that problem?

# Gallagher (Least-Squares) Index

$$\sqrt{\frac{\sum_i (v_i - s_i)^2}{2}}$$

- This index penalizes large disproportionalities more than small ones
- It still suffers from paradoxes (as does every index that is based on absolute, rather than relative, vote-seat differences)

# Sainte-Laguë Index

- Sainte-Laguë independently derived his formula (which is equivalent to Webster's method) by seeking to minimize the following index of proportionality
- Let $S_i$ be the number of seats $i$ gets and $V_i$ the number of votes
- Let TS=total number of seats and TV=total number of votes

$$\sum_i V_i \left( \frac{S_i}{V_i} - \frac{TS}{TV} \right)^2$$

- What do $\frac{S_i}{V_i}$ and $\frac{TS}{TV}$ represent?

## Two-tier districting

- It's commonly acknowledged that there is a trade-off between proportionality and constituency service
  - Because there is a direct link between proportionality and DM

  Two-tier systems compensate by electing part of the legislature at large (or from a very large region), and part from smaller districts (the "lower tier")
- Used by most largest remainder systems and modified Sainte Laguë systems

## How two-tier districting works

- For highest average systems, a proportion of the total number of seats are set aside for the second (e.g. national) tier
  - After seats are allocated at the regional level, these remaining seats are used to "top up" parties who did not receive their fair (proportional) share of the district seats
- For largest remainder systems the % of second tier seats is not fixed, and equal the remainder seats that were left over after the first round of seat allocation in the regional districts

# In-class question

Suppose that there are two districts holding elections, using largest remainder with Droop quota ($\frac{\text{votes}}{\text{seats+1}} + 1$ with remainder dropped). In each district there are 100 votes total, 9 seats.

| | District 1 | | District 2 |
|---|---|---|---|
| Party 1 | 54 | | 32 |
| Party 2 | 40 | | 42 |
| Party 3 | 6 | | 26 |

(1) In each district, allocate seats in 1 round according to Droop quota but don't allocate using the remainders

(2) Now pool the vote totals across districts and allocate 18 seats using Droop and remainders (Droop =11)

(3) How many "upper tier" (compensatory) seats does each party deserve? How does this compare to allocations without compensation?

## Things to think about for your lab this week

- In 2012 Democrats received 1.4 million more votes for the House of Representatives than Republicans, but Republicans won the House by 234 to 201
- What factors might have contributed to this disproportionality?
- How can we disentangle these various possible causes?