Fair Division in Theory and Practice

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Lecture 3: Apportionment

Fair representation

- We would like to allocate seats proportionally to the 50 states
  - States deserve fraction of seats equal to their fraction of total U.S. population
- Unfortunately, states must receive an integer number of seats; “ideal” entitlements generally not whole numbers
  - So we have to deal with a rounding problem
  - This seems surmountable

The Constitution specifies that...

- “Representatives shall be apportioned among the several States ... according to their numbers ...”
- “The number of Representatives shall not exceed one for every thirty thousand, but each State shall have at least one ...”
  - A minimum size requirement for districts
- Constitutionally, there can be no less than 50 representatives, and no more than approx. 10,000

Two main historical issues

1. How populous a district should be
   (ie. how large the House should be)
   - After 1910 census House increased to 435 members, where it was capped by law in 1941
   - Current proposals to increase House size: "Wyoming plan," Clemons v. Dept of Commerce: NV had 960K while WY had 523K
2. How to treat fractional entitlements to Representatives
   - Ideal size of districts=US population/435 (*standard divisor*)
   - Today, this would be about 726,000
   - A state's ideal number of representatives= $\frac{\text{State pop.}}{\text{U.S. pop.}} * 435$

First process of apportionment occurred in 1794, using 1790 census figures. 105 seats needed to be allocated to 15 states.

- The total population was 3,893,874
- The *Standard Divisor* is the "ideal" number of people who should be represented by a single seat
- In 1794 Standard Divisor $= \frac{3{,}893{,}874}{105} = 37{,}084.51$
- To calculate number of seats a state should get, we divide state's population by this number

CT	237,655	6.41
DE	59,096	1.59
GA	82,548	2.22
KY	73,677	1.98
MD	319,728	8.62
MA	475,199	12.81
NH	141,899	3.82
NJ	184,139	4.96
NY	340,241	9.17
NC	395,005	10.65
PA	433,611	11.69
RI	69,112	1.86
SC	249,073	6.71
VT	85,341	2.30
VA	747,550	20.15

CT	237,655	6.41	6
DE	59,096	1.59	1
GA	82,548	2.22	2
KY	73,677	1.98	1
MD	319,728	8.62	8
MA	475,199	12.81	12
NH	141,899	3.82	3
NJ	184,139	4.96	4
NY	340,241	9.17	9
NC	395,005	10.65	10
PA	433,611	11.69	11
RI	69,112	1.86	1
SC	249,073	6.71	6
VT	85,341	2.30	2
VA	747,550	20.15	20
Total whole seats allocated:			96

CT	237,655	6.41	6
DE	59,096	1.59	2
GA	82,548	2.22	2
KY	73,677	1.98	2
MD	319,728	8.62	9
MA	475,199	12.81	13
NH	141,899	3.82	4
NJ	184,139	4.96	5
NY	340,241	9.17	9
NC	395,005	10.65	11
PA	433,611	11.69	12
RI	69,112	1.86	2
SC	249,073	6.71	7
VT	85,341	2.30	2
VA	747,550	20.15	20

Total if rounding at .5:	106

CT	237,655	6.41	6
DE	59,096	1.59	1
GA	82,548	2.22	2
KY	73,677	**1.98**	2
MD	319,728	**8.62**	9
MA	475,199	**12.81**	13
NH	141,899	**3.82**	4
NJ	184,139	**4.96**	5
NY	340,241	9.17	9
NC	395,005	**10.65**	11
PA	433,611	**11.69**	12
RI	69,112	**1.86**	2
SC	249,073	**6.71**	7
VT	85,341	2.30	2
VA	747,550	20.15	20
Hamilton: award to largest remainders			105

- Hamilton proposed allocating whole number of seats in first round and then awarding remaining seats to states with largest remainders
- The method passed Congress, however President Washington vetoed it – the first presidential veto in U.S. history.
- Why?

Hamilton’s method treats fractional entitlements differently from election to election (the same fractional entitlement could be rounded either up or down from one election to the next)

Moreover, Virginia did not do well under this plan.

CT	237,655	6.41	6
DE	59,096	1.59	1
GA	82,548	2.22	2
KY	73,677	**1.98**	2
MD	319,728	**8.62**	9
MA	475,199	**12.81**	13
NH	141,899	**3.82**	4
NJ	184,139	**4.96**	5
NY	340,241	9.17	9
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PA	433,611	**11.69**	12
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VT	85,341	2.30	2
VA	747,550	20.15	20

Hamilton: award to largest remainders	105

After Washington vetoed Hamilton's method, Congress did not have enough votes to override

Switched to method proposed by Jefferson – called a *divisor method*

Divisor methods divide all populations by a common number and round fractions in a predetermined way; the divisor is adjusted until the exact number of seats are allocated

Divisor Methods

- Calculate each state's ideal number of seats: $\frac{\text{State pop.}}{\text{U.S. pop.}} * 435$
- Fix *any* rounding rule
  - Round all numbers down (Jefferson's method)
  - Round all numbers up
  - Round all numbers down if fraction $< .5$, up otherwise
  - Round all numbers at the geometric mean: for any number in (a,b) interval, round at $\sqrt{a * b}$
- If rounding rule doesn't result in correct number of seats being allocated, keep adjusting divisor until the correct number of seats are allocated

Jefferson's method

1. Calculate the standard divisor:

$$\frac{\text{Total U.S. population}}{\text{Total number seats}}$$

2. Compute the deserved number of seats per state:

$$\frac{\text{State population}}{\text{Standard divisor}}$$

   Round each of these numbers *down*

3. If these rounded numbers sum to the total number of seats, we are done

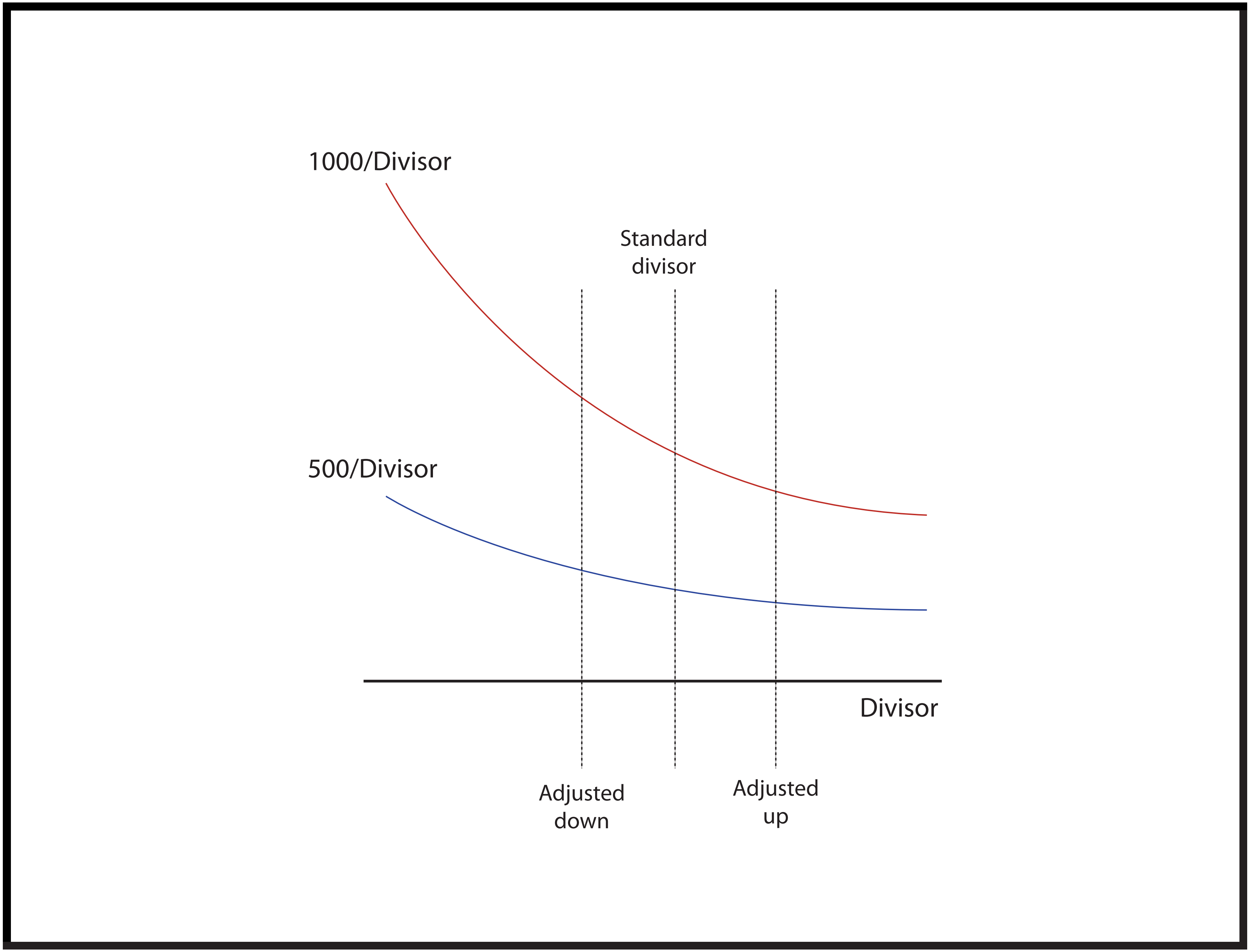
4. Otherwise, replace the standard divisor with a different number and repeat the process. Keep repeating until the number of seats allocated equals the correct number of seats

Jefferson's method is computationally intensive because it involves trial and error until a divisor is found that works

Jefferson's method was used to apportion seats in 1794; only change that would result in our previous table (compared to Hamilton's) is that RI loses a seat and VA gains a seat

Why Jefferson's method advantages large states

- The modified divisor is always smaller than the standard divisor, because the method rounds all seat entitlements down
- Lowering the divisor causes larger states to have a *faster* increase in their (modified) standard quotas
- Thus, states with larger populations have a greater chance of increasing their quota over the next whole number
- This can be extreme; a very large state could possibly increase its standard quota over two whole numbers before another state gets a seat


1000/Divisor
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In 1822 a strange problem was revealed in Jefferson's method

- NY's population was 1,368,775
- US population was 8,969,878, and 213 seats were to be apportioned
- NY's ideal number of seats was 32.5; under Jefferson's method it received 34 seats
- In 1832 the same thing happened; NY's ideal number of seats was 38.5, but it received 40 seats

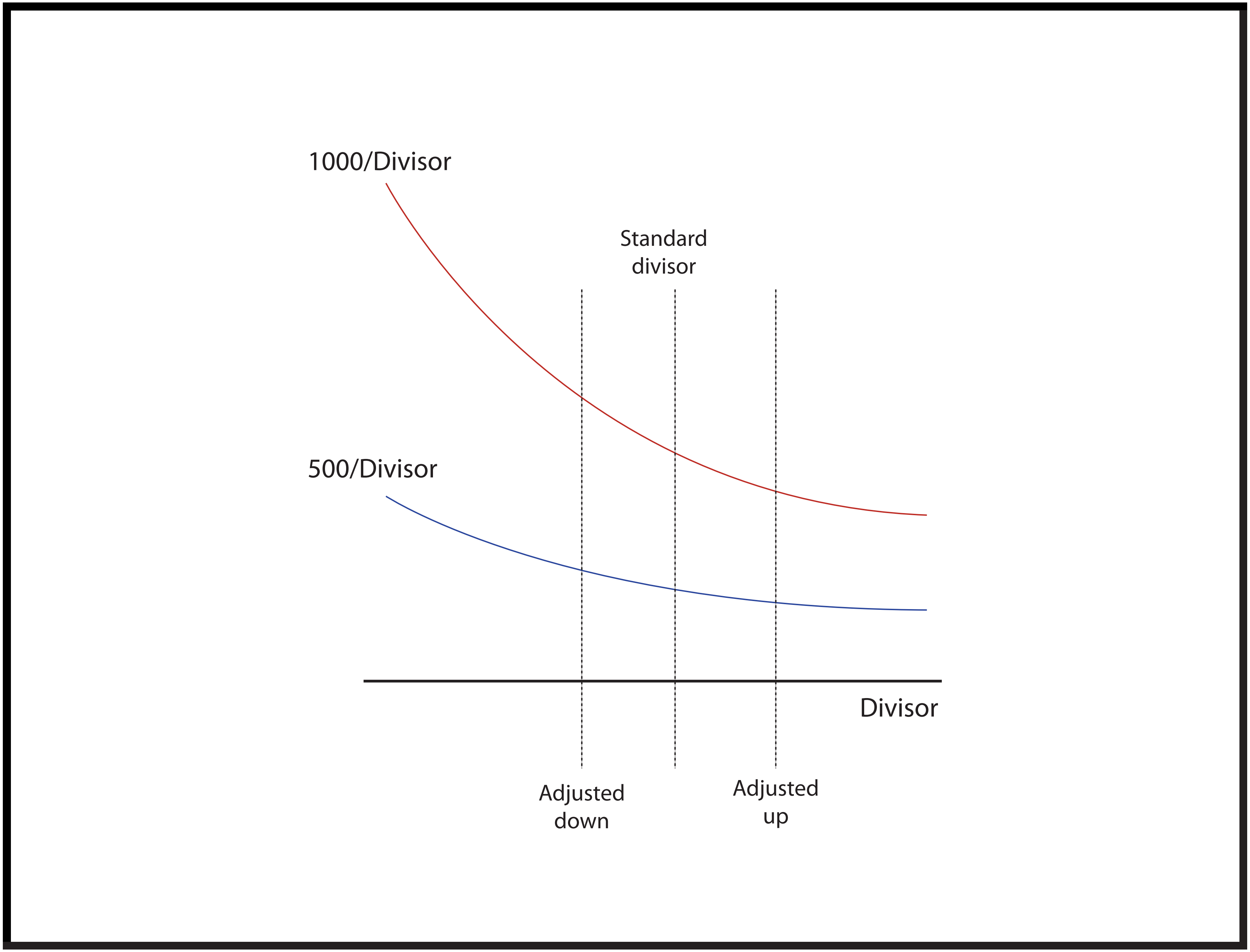
Jefferson's method, ***and every divisor method***, violates the ***quota rule***

Quota Rule:

- Suppose a state deserves a "fair share" of seats equal to q (which in all likelihood is not a whole number)
- That state should always receive either q rounded up, or q rounded down
  - For example, suppose New York deserves 32.25 seats
  - NY should then receive either 32 or 33 seats
  - If it gets 31 or fewer seats, or 34 or more seats, then the rule *violates quota*

Immediately several new districting plans were proposed

- Adams proposed a method identical to Jefferson's, except that quotas would be rounded *up* instead of *down*
- Starting with standard divisor, all standard quotas would be rounded up
- How would standard divisor have to be adjusted to ensure correct number of seats are distributed?
- Do you see a problem with this method (in terms of advantaging one kind of state over another)?


1000/Divisor
Standard
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Webster's method (used in 1842)

- Same as Jefferson and Adams's but rounds at .5 instead of always rounding down or up
- Standard divisor could be modified either up or down, depending on whether too many or too few seats were initially allocated
- Neutral in how it treats large and small states
  - If the standard divisor allocates too few seats, it slightly advantages large states
  - If it allocates too many seats it slightly advantages small states
- It also violates the quota rule, but does so very rarely (would not have happened in a single apportionment since 1794; Jefferson would have violated quota rule in *every* apportionment)

Because Webster's method *can* violate quota, people remained skeptical

In 1850 Congressman Vinton proposed a "brand new" apportionment method – it was Hamiton's method

- It can never violate the quota rule – why?

In 1852 Congress passed a law adopting Hamilton's method, and it was essentially used for next 50 years

When they adopted Hamilton's method they also passed another bill

- In 1852 and thereafter they would always adjust the size of the House to a number such that both Hamilton's and Webster's methods would yield the same distribution of seats

In 1872 Congress illegally, and in direct violation of Constitution, passed an apportionment plan that wasn't based on any formula at all

- Based on this plan, Hayes defeated Tilden in the 1876 election – if seats (and consequently electoral votes) had been distributed according to Hamilton or Webster's methods, Tilden would have easily won
- Congress switched back to Hamilton's method in 1882

From 1850–1900 U.S. population grew from 23 million to 75 million

- 14 states were added
- Internal migration: urban population increased from 15% to 40%
- House size grew from 234 (1856) to 386 (1902)
- Shifts in state power / populations subjected Hamilton formula to a variety of trials that ultimately led to its abandonment

In 1882, Census Bureau supplied Congress with a list showing apportionment results produced by Hamilton's method for all House sizes between 275 and 350

- As the size of the House increased from 299 to 300, Alabama lost a seat
- The Alabama Paradox: Increasing the number of seats to be allocated causes a state to lose a seat
- Is a very common occurrence with Hamilton's method

Alabama Paradox under Hamilton's method, 1880

State	Quota at 299	Seats Awarded	Quota at 300	Seats awarded
Alabama	7.646	8	7.671	7
Texas	9.640	9	9.672	10
Illinois	18.640	18	18.702	19

Ultimately Congress chose a House size of 325, a number at which the paradox did not occur

In 1902 things got much worse

- Census Bureau distributed table showing allocations of 350–400 seats
- For House sizes of 350–382, 386, 389–90, Maine got 3 seats
- For all other House sizes between 350 and 400, Maine got 4 seats
- For House size of 357, Colorado got two seats; for all other sizes between 350 and 400 it got three
- A bill was presented to make the size of the House precisely 357 by Census Committee chair Albert Hopkins, (Illinois)

"Maine loses on 382. She loses again with 386, and does not lose with 387 or 388. Then she loses again on 389 and 390, and then ceases to lose. Not only is Maine subjected to the assaults of the chairman of this Committee, but it does seem as though mathematics and science had combined to make a shuttlecock and battledore of the State of Maine in connection with the scientific basis upon which this bill is presented...In Maine comes and out Maine goes...God help the State of Maine when mathematics reach for her and undertake to strike her down."

— Rep. John Littlefield (Maine)

"It is true that under the majority bill Maine is entitled to only three seats, and ... the seat of the gentleman who addressed the House on Saturday last is the one in danger. In making this statement he takes a modest way to tell the House and the country how dependent the State of Maine is upon him. How delightfully encouraging it must be to his colleagues of that State to know the esteem in which they are held by him. Maine crippled! ... That great State crippled by the loss of LITTLEFIELD! Why, Mr. Speaker, if the gentleman's statement be true that Maine is to be crippled by this loss, then I can see much force in the prayer he uttered here when he said 'God help the State of Maine.' "

— Rep. Albert Hopkins (Illinois)

Ultimately, proposal led to such heated debate that Webster's divisor method was used that year & Hamilton's method was abandoned (Maine got its 4th seat)

Hamilton's method also suffers from the *population paradox*

- Following a census, a state should not *gain* population and *lose* a seat, while another state *loses* population and *gains* a seat

and the *new states paradox*

- The addition of a new state (and an increase in seats for that state) causes seat allocations to change in other states

Why do these paradoxes happen under Hamilton's method?

State	Pop	Std.Quota	Integer	Fraction	Final
A	1320	1.32	1	.32	2
B	2310	2.315	2	.315	2
C	3140	3.135	3	.135	3
D	3230	3.230	3	.23	3
Total	10,000	10	9	1	10

State	Pop	Std.Quota	Integer	Fraction	Final
A	1320	1.452	1	.452	1
B	2310	2.541	2	.541	3
C	3140	3.454	3	.454	3
D	3230	3.553	3	.553	4
Total	10,000	11	9	2	11

Why does the Alabama paradox happen?

- For 10 seats, A's exact apportionment was 1.320 seats, because A had 13.2% of the total population.
- When 1 extra seat is added, what % of that seat does A deserve?

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- Therefore A deserves what exact # of seats?
  - $1.320 + .132 = 1.452$ seats
  - (We add .132 to A's former total)
- A larger state, say one that deserved 3.230 seats when 10 were available now deserves how many?

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  - $1.320 + .132 = 1.452$ seats
  - (We add .132 to A's former total)
- A larger state, say one that deserved 3.230 seats when 10 were available now deserves how many?
  - $3.230 + .323 = 3.553$
- The fractional part of the larger state's seat leapfrogged over A's fractional part.

Point: history & common sense give us a set of principles for what fair apportionment methods should and should not do

1. They should not take a seat away from a state that has gained population and give it to one that has lost population
2. They should not take seats away from any state when populations are unchanged and more seats are added to the House
3. They should allocate seats as closely as possible to each state's "fair share," either rounding that share up or down

The Balinski-Young Theorem: Any system that satisfies (3) *must* violate both (1) and (2), and vice versa.

Webster's method used again in 1912

At this time Hill (chief statistician of the Census Bureau) proposed a new method

- While Webster's method rounds at .5 (or the arithmetic mean of two consecutive numbers), Hill proposed to round at the geometric mean of consecutive numbers
- The geometric mean of a and b is $\sqrt{a * b}$
- For the quota 8.49, Hamilton's method would round down
- The geometric mean of 8 and 9 is 8.485; Hill's method would round *up*

- In 1922 debate between Hill's method and Webster's method ended with no new apportionment plan
- In 1932 both plans yielded the same result, so a decision between the two did not have to be made
- In 1942 Hill's method gave an extra seat to Arkansas at the expense of Michigan; since Arkansas was Democratic and Michigan Republican, Hill's method passed Congress
- FDR signed it into law; it's been used since

In 1990 Montana lost a congressional seat and sued the Dept. of Commerce (home of the Census Bureau); argued that Hill's method violated one person, one vote

District court agreed – ruled that gvt could not reapportion House using that method

Went to the Supreme Court who overturned ruling, arguing that there are many ways of measuring equality

Current controversies

Montana contended the Dean method (rounding at the harmonic mean) minimizes difference between largest and smallest district sizes.

One of federal government's counterarguments was that this is a pairwise test; Dean method does not minimize such differences when all states are considered simultaneously.

The federal government proposed variance as a means of testing apportionment formulas against various criteria of fairness.

How to evaluate fairness?

When variance in district size was considered in 1990, the variance under the Hill method was the smallest of any of the apportionment methods considered.

Variances can be calculated for many criteria of fairness, however.

If criterion considered is individual share of a Representative, the Hamilton method minimizes differences in these shares, with Webster the best of the divisor methods.

If we care about fairness in our procedures, what should we do?

- It is mathematically impossible for any procedure to satisfy every common-sense principle of fairness (or even most of these principles)
  - Even in a simple decision environment (e.g. apportionment) problems are inescapable
- The US apportionment story demonstrates that there is no best, or most fair, apportionment system
- There is a "rational basis" for each of these formulas