Fair Divsion in Theory and Practice

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Lecture 2: Moving Knives

Can two people agree on what is exactly $\frac{1}{2}$ of a piece of cake?

- Divide and choose favors the chooser
- It is not necessarily equitable
- If both parties could agree on a piece that is exactly $\frac{1}{2}$ value
  - The other piece must also be of $\frac{1}{2}$ value
  - Because both parties must see:

$$1 - \frac{1}{2} = \frac{1}{2}$$

  - Thus, an equitable solution would be obtained
- But it is not necessarily Pareto efficient
  - Can you devise an example to show this?

Overview

- Moving knife algorithm
  - Two people
  - More people
- Two-party agreement on $\frac{1}{2}$ of a cake

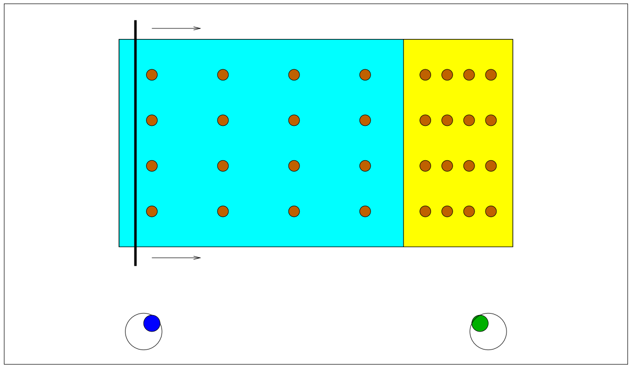
Moving Knife Algorithm $n = 2$ (Dubins and Spanier, 1961)

Procedure:

- Players P_1 and P_2 observe a knife moving slowly across the cake, from left to right.
- Whoever among $\{ P_1, P_2 \}$ calls *stop* gets the piece to the left.
- The other player gets the piece to the right.

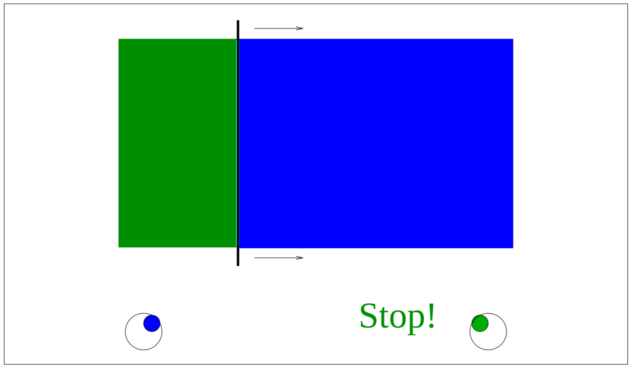
Moving Knife Algorithm $n = 2$

The players observe the knife moving across the cake, from left to right.



One player says “stop” and is given the piece of cake to the left of the knife. The other player gets the other piece.

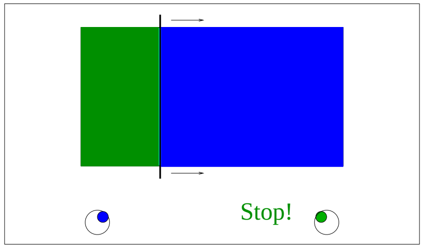
Here, suppose the green player calls a halt. That player receives the green portion of the cake, and the other player, who said nothing, receives the blue portion of the cake.



This is the same outcome as Divide and Choose, except that the cutter acts *endogenously*.

Moving Knife Algorithm $n = 2$ (Stategy)

Define the pieces left and right of the knife as π_L and π_R, respectively. To obtain a proportional outcome, P_i must say "stop" when $v_i(\pi_L) = \frac{1}{2}$



Theorem 1 *If player P_i fails to say stop when $v_i(\pi_L) = \frac{1}{2}$, then P_i can receive less than $\frac{1}{2}$ of the cake.*

Proof: Two cases:

- If P_i said stop too early, when $v_i(\pi_L) < \frac{1}{2}$, then P_i certainly gets less than $\frac{1}{2}$ the cake.
- If P_i delays saying stop, so that $v_i(\pi_L) > \frac{1}{2}$, then the other player can say stop before P_i. Then P_i receives π_R, $v_i(\pi_R) = 1 - v_i(\pi_L) < \frac{1}{2}$.

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Moving Knife Algorithm $n = 2$

- This is called a *continuous* algorithm in political science literature.
  - Why is it called continuous?
  - What problems could arise for implementation?
  - How would you address those problems?
- Divide and Choose did not generalize beyond $n = 2$, but the Dubins Spanier algorithm does!

Moving Knife Algorithm, arbitrary n

Procedure:

- A knife is moved slowly across the cake, from left to right.
- Any player without an assigned piece can say "stop".
- That player is assigned the piece to the left of the knife.
- When only one player remains, that player is assigned the piece to the right of the knife (π_R).

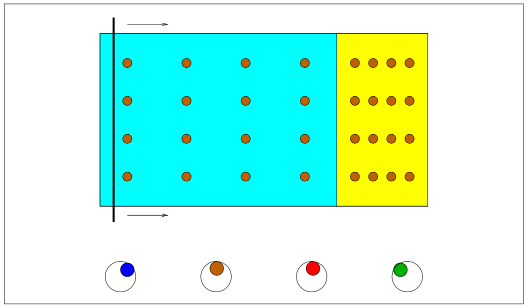
Strategy: If π_L is the cake currently to the left of the moving knife, then any player P_i must say stop when

$$v_i(\pi_L) = \frac{1}{n}$$

to guarantee P_i a proportional piece of the cake.

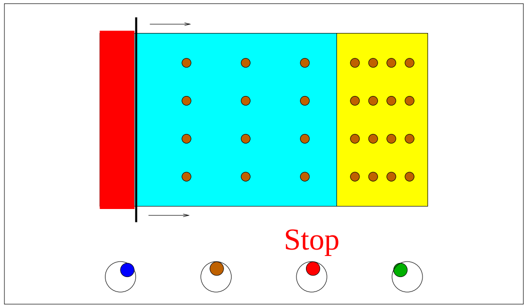
Moving Knife Algorithm, arbitrary n

The knife starts to move across the cake, from left to right. There are four players in this example, distinguished by the color of their eyeballs.

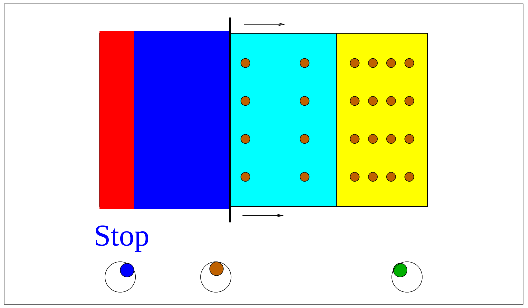


Suppose the red player is the first to say "stop". That player is assigned the piece of cake to the left of the knife, also shown in red. Because red called stop first:

$$\forall\ i \neq \text{red}\ v_i(\text{red piece}) < \frac{1}{4}$$

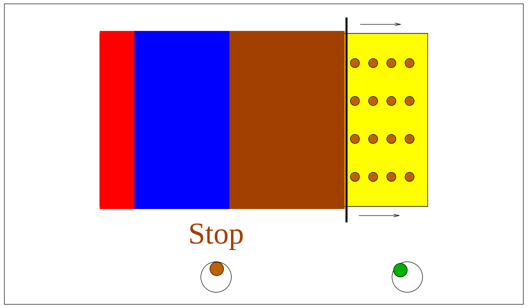


The red player drops out of the algorithm. Suppose blue is the next to say "stop". The blue piece is assigned as shown.

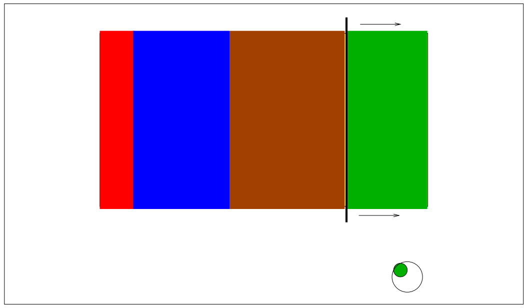


Moving Knife Algorithm, arbitrary n

Two players remain, brown and green. If brown calls stop first, then the brown piece is assigned as shown.



The green player never called stop, so as the last player, the green player receives the piece to the right of the stopped knife.



Moving Knife Algorithm, arbitrary n

- Is there always enough cake for everybody?
  - When done, is the following true?

$$\forall\ i\ v_i(\pi_i) \geq \frac{1}{n}$$

- What happens if some P_i tries to be greedy and does not say stop the first time $v_i(\pi_L) = \frac{1}{n}$?
- Can envy develop? If so can it exist between any players, or only between some?
  - Hint: Can the player who receives π_R envy anybody?
- Is the result necessarily equitable?
- Is the result necessarily stable?

Two people agree on $\frac{1}{2}$ a cake
Austin's Algorithm

Overview P_1 and P_2 seek agreement on an exact division of cake into two equal-valued "pieces". Preferences are secret.

Procedure

- A knife moves across the cake from left to right.
- At some point, one player P_k says "stop"
- P_k is given a second knife, which is positioned at the extreme left side of the cake.
- P_k moves both knives to the right.
- At some point, the other player (P_{3-k}) says stop.
- The two knives cut the cake where they are stopped. The piece between the knives is given randomly to one player, the rest of the cake to the other.

Two people agree on $\frac{1}{2}$ a cake

Strategy (working backwards)

- Because P_k controls the knives when the cake is finally cut, and the knives can be stopped at any time by the other player, P_k must always ensure that the cake between the knives (π_{in}) and outside of the knives (π_{out}) has the same value:

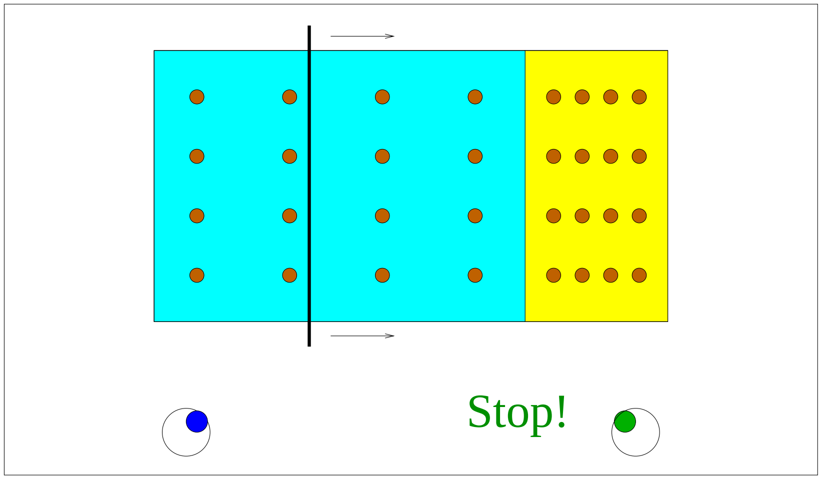
$$v_k(\pi_{in}) = v_k(\pi_{out}) = \frac{1}{2}$$

- Before that, after P_k took both knives, the other player could say stop at any time, including right away. Thus, whoever takes the second knife must see the first knife's position as bisecting the cake's value.

Two people agree on $\frac{1}{2}$ a cake

- The blue green players observe the knife moving across the cake.
- When one believes the knife has reached a point where the cake's value is bisected, that player will say "stop".
- Here, the green player says "stop".
- If we call the pieces left and right of the knife π_L and π_R, respectively, then we have

$$v_{\text{green}}(L) = v_{\text{green}}(R) = \frac{1}{2}$$



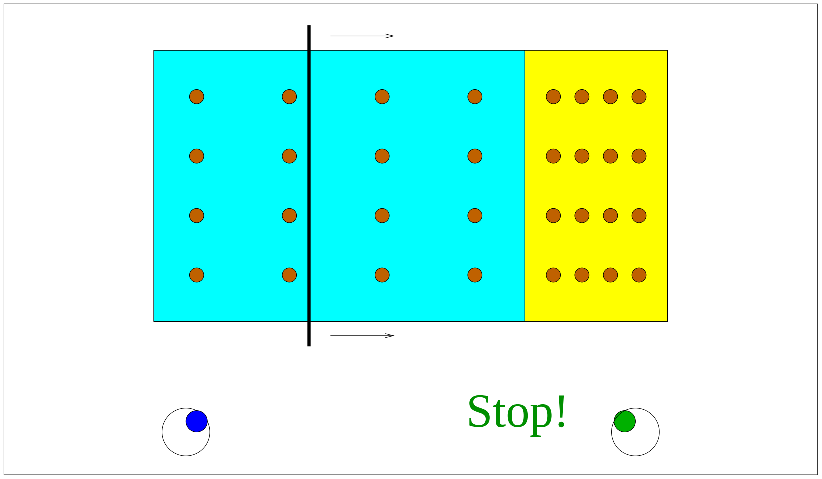
Two people agree on $\frac{1}{2}$ a cake

- If we call the pieces left and right of the knife π_L and π_R, respectively, then we have

  $$v_{\text{green}}(L) = v_{\text{green}}(R) = \tfrac{1}{2}$$

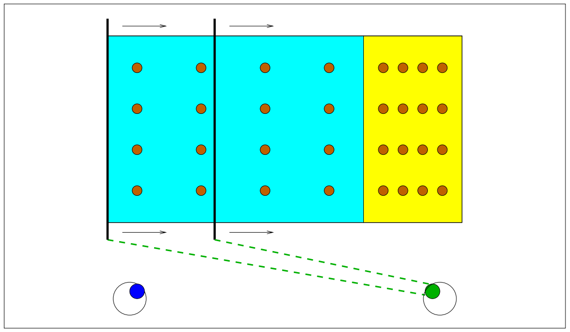
- But what about the blue player?
  - Because blue did *not* say "stop", this must mean:

    $$\begin{array}{lcl} v_{\text{blue}}(L) & < & \tfrac{1}{2} \\ v_{\text{blue}}(R) & > & \tfrac{1}{2} \end{array}$$

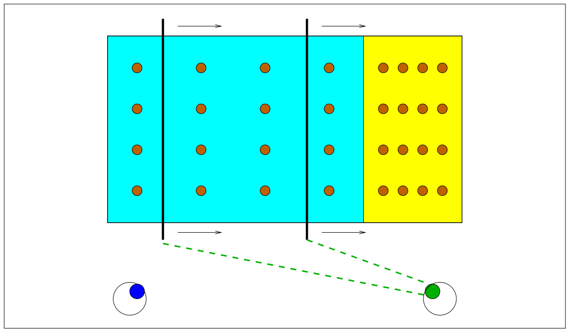


Two people agree on $\frac{1}{2}$ a cake

The green player is given a second knife, which starts at the extreme left end of the cake.



The green player continues to move both knives to the right, always keeping $\frac{1}{2}$ of the cake's value between the knives, according to green's view.



If π_{in} and π_{out} represent the cake between and outside of the knives, respectively, then we have at all times:

$$v_{\text{green}}(in) = v_{\text{green}}(out) = \frac{1}{2}$$

Two people agree on $\frac{1}{2}$ a cake

Eventually, the value between the knives reaches $\frac{1}{2}$ in blue's view, and blue says "stop".

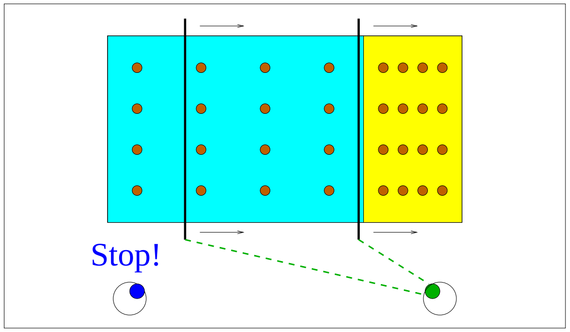
At that point:

$$v_{\text{green}}(in) = \tfrac{1}{2}$$
$$v_{\text{green}}(out) = \tfrac{1}{2}$$

because green always moved the knives to maintain that condition, and

$$v_{\text{blue}}(in) = \tfrac{1}{2}$$
$$v_{\text{blue}}(out) = \tfrac{1}{2}$$

because blue said "stop".

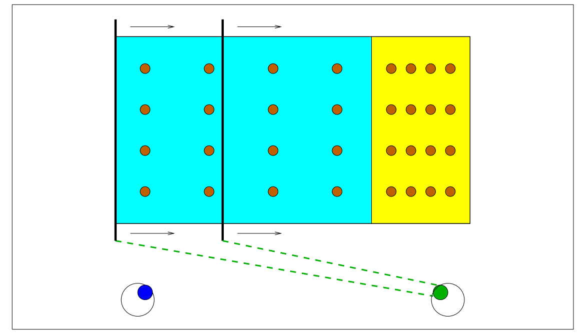


How do we know blue will always be able to say "stop"?

Two people agree on $\frac{1}{2}$ a cake

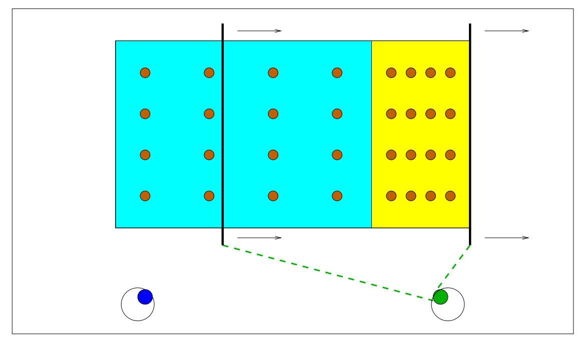
When green received the second knife, we had:

$$\begin{aligned} v_{\text{green}}(in) &= \tfrac{1}{2} \\ v_{\text{green}}(out) &= \tfrac{1}{2} \\ v_{\text{blue}}(in) &< \tfrac{1}{2} \\ v_{\text{blue}}(out) &> \tfrac{1}{2} \end{aligned}$$



If green moved the knives as far right as possible, we would get the picture shown here, for which we have:

$$\begin{aligned} v_{\text{green}}(in) &= \tfrac{1}{2} \\ v_{\text{green}}(out) &= \tfrac{1}{2} \\ v_{\text{blue}}(in) &> \tfrac{1}{2} \\ v_{\text{blue}}(out) &< \tfrac{1}{2} \end{aligned}$$



By the intermediate value theorem, somewhere in between $v_{\text{blue}}(in)$ must move from $< \frac{1}{2}$ to $> \frac{1}{2}$, so $v_{\text{blue}}(in)$ reaches exactly the value $\frac{1}{2}$.

Questions

- Devise an algorithm so that two parties can agree on a piece of cake whose value is exactly $\frac{1}{2^i}, i \geq 1$.
- Devise an algorithm so that two parties can agree on a piece of cake whose value is exactly $\frac{1}{k}, k \geq 1$.
- Devise an algorithm that divides a cake into m pieces, each of value $\frac{1}{m}$ to both parties.