

# Fair Division in Theory and Practice

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*Lecture 2b: The Adjusted Winner Procedure*

## How do we share things that are indivisible?

*Annie and Ben get a divorce*

1. House: close to Annie's office & she designed the new kitchen.
2. Investment Acct: Combined life savings; important to both people.
3. Baby grand piano: Annie takes lessons, but Ben is the skilled pianist and it is his most prized possession.
4. Plasma TV: Ben chose it and watches more TV than Annie. He also uses it to screen movies and writes reviews for the local paper.
5. Tawny, the golden retriever: Goes to work with Annie most days. Annie very attached.
6. Car: Annie walks to work, so somewhat less valuable to her.

The *adjusted winner* procedure: envy-free, equitable, efficient procedure for 2 people to divide multiple, potentially indivisible, things

- **Envy-free:** No one wants another person's bundle of things
- **Efficient:** There's no other allocation that makes one person strictly better off while not making anyone else worse off
- **Equitable:** Everyone receives exactly the same percentage of his or her total value of the good(s)

Each person gets 100 points to allocate among items. Annie and Ben's distributions are:

Annie	Item	Ben
35	House	15
20	Investments	25
10	Piano	25
5	TV	15
25	Tawny	10
5	Car	10

## How the procedure works

**Round 1:** Each person is initially allocated the items that they *strictly* value higher than the other person. Next, tied items awarded, one at a time, to person with fewest points. (If point totals also tied, they are awarded to either person.)

**Round 2:** If the point totals are equal, the procedure is done. Otherwise proceed to next step.

**Round 3:** The person with most points is the *initial winner*. Calculate the ratio of initial winner points to loser's points and order the items from lowest ratio to highest (ratio always  $\geq 1$ ). From lowest to highest, transfer items (*or fractions of items*) from initial winner to loser until point totals equal.

How would we divide these things?

Annie	Item	Ben
35	House	15
20	Investments	25
10	Piano	25
5	TV	15
25	Tawny	10
5	Car	10

Ben gets more points; we calculate preference ratios for the things he won:

$$\text{Investments} = 25/20=1.25$$

$$\text{Piano}=25/10=2.5$$

$$\text{TV}=15/5=3$$

$$\text{Car}=10/5=2$$

We need to calculate Annie's share of investments. Let  $x$  be this share. Equitability tells us that we want

$$60 + 20x = 50 + 25(1 - x)$$

which gives us  $x = \frac{1}{3}$ .

We wind up with Annie getting house, Tawney and  $\frac{1}{3}$  of investment portfolio. Ben gets  $\frac{2}{3}$  of portfolio, TV, piano and car. Each receives  $66\frac{2}{3}$  points.

- Procedure is equitable (obvious)
- It's efficient (proof is not particularly easy)
- It's envy-free. Why? Remember with 2 people proportional  $\Rightarrow$  envy-free. Suppose it's not proportional. By equitable, this implies both get less than  $\frac{1}{2}$ . But this violates efficient (they could switch bundles and both be better off). Thus, it's proportional, and so envy-free.

## Our first impossibility result

With three or more people, it is impossible to design a procedure that is guaranteed to satisfy all three fairness criteria: equitability, envy-freeness and efficiency.

## Points to note

- Only one item is ever split
- The procedure can be modified (e.g. a 60/40 prenup with respect to property division)
- Do you think this procedure would be easy to implement in practice?

## Manipulability of these procedures

Annie & Ben are getting a divorce and share the following:

Annie	Item	Ben
50	Townhouse in CWE	30
20	Cardinals Tickets	50
30	Matisse painting	20

What's the final distribution?

Annie knows Ben fairly well, and thinks she can estimate his valuations. What happens if she submits these “fake” valuations?

Annie	Item	Ben
32	Townhouse in CWE	30
48	Cardinals Tickets	50
20	Matisse painting	20

Ben can also do better by submitting the following fake preferences (provided Annie is being honest).

Annie	Item	Ben
50	Townhouse in CWE	45
20	Cardinals Tickets	25
30	Matisse painting	30

## What if both are dishonest?

True values	Annie	Item	Ben
	50	Townhouse in CWE	30
	20	Cardinals Tickets	50
	30	Matisse painting	20

Fake values	Annie	Item	Ben
	32	Townhouse in CWE	45
	48	Cardinals Tickets	25
	20	Matisse painting	30

## Adjusted winners and the Middle East

(Massoud, *Journal of Conflict Resolution*, 2000); lays out 5 key areas of disagreement

1. West Bank: Israelis don't want to leave their homes; Palestinians believe settlements are illegal.
2. East Jerusalem: In 1967 Israel unified control over Jerusalem in Six Days War. Majority of residents of East Jerusalem are Palestinian; both sides argue East Jerusalem central to their sovereignty.
3. Palestinian Refugees: Israel has refused to recognize that its establishment and expansion in 1948/1967 displaced Palestinian communities. Palestinians insist Israel recognize refugees' "right of return" to Israel and compensate Arab states that hosted refugees.

4. Palestinian sovereignty

5. Security: Some Israelis fear terrorism would flourish under a Palestinian state that lacks the means to fight terrorism. Specific issues include border control, airspace, security in Jerusalem, “early warning stations” in West Bank and Gaza to warn against surprise attack

Massoud (political scientist) looked at expert opinions, interim agreements, working plans to come up with point estimates of both sides

Israel	Item	Palestine
22	West Bank	21
25	East Jerusalem	23
12	Palestinian refugees	18
15	Palestinian sovereignty	24
26	Security	14

How are the issues allocated?

Israelis get East Jerusalem, security and  $2/7$  of West Bank.

Palestinians get refugees, sovereignty and  $5/7$  of West Bank.

What do you think about this project as a real-world instantiation of the kinds of cake-cutting algorithms we have been studying?

## Fun Fact

It was known in the 1940's that an envy free procedure definitely exists for any number of people.

But it wasn't until 1960 that an envy-free procedure was actually found for 3 people (the *Selfridge-Conway* procedure, although it does not generalize to 4 people).

Envy-free procedures typically do not generalize to larger numbers of people; they get significantly more complicated as the number of people increases.