Fair Divsion in Theory and Practice

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Lecture 1: Introduction

Some things to watch for on these slides

Definition 1 definition: *A concept that will recur, with which you should become familiar*

If you see something red like this, it is a question or issue that you may see again on an exam. These will be treated as follows:

- a pause in lecture for you to address the issue or solve the problem
- homework assignments
- topics for discussion on piazza
- topics for further investigation in class or lab

Our first problem: divide "cake" fairly

divide share, split, partition, assign

cake metaphor for divisible goods

> **Definition 2** divisible*: total value is conserved when cake is divided*

fairly Aha! What do we mean by this?

- "That's not fair!" Have you said this? What did you mean by it? For our first two children:
  - Determining a "winner"
  - Is a random process necessarily fair?
    - Rosencrantz and Guildenstern!
  - Not the same thing as sharing something
- If there are different notions of fairness, how do we determine which one to choose?

Key: Participating parties must agree on fairness criteria

Fairness in literature

- What happens when unfairness is perceived?
  - Cain and Abel over an offering

    If you offer properly, but *divide improperly*, have you not sinned?[a]

  - Jacob and Esau over birthright
  - *The Honeymooners.* See video @2:30: The *process* can be more important than the *outcome*
- Jacob, Leah, and Rachel: Jacob *vs.* Laban's notions of fair treatment
- Divide or not divide?
  - King Solomon and the baby

[a] English translation from the Greek *Septaugint*, an early Greek translation of the Hebrew Bible.

– Recast in *Seinfeld*'s bicycle

- Zeus and Prometheus share a steak[b]

[b] Hesiod's *Theogony*, 700 BC

Properties of Fair Division Problems

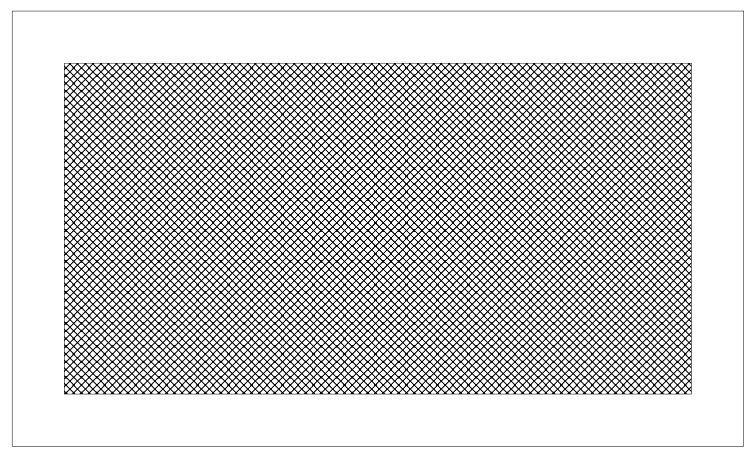
- Divisible or indivisible goods?
- How many parties or participants?
- How to the parties value portions of the goods?
- What do we mean by “fair”?
- How much time are we willing to spend on a solution?

Alice and Bob share cake

- We assume cake has unit value to each party.
- Problems in fair division are interesting only when parties view the cake differently.

If all parties value the cake uniformly and identically, then fair division reduces to the geometric problem of determining exactly equal slices of the cake by area or volume.

Given n participants, each receives $\frac{1}{n}$ of the cake. The particular region of the cake obtained by a given participant does not matter.



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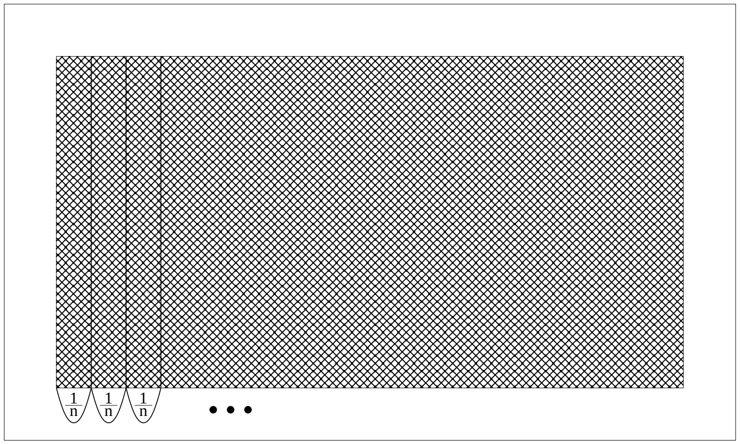
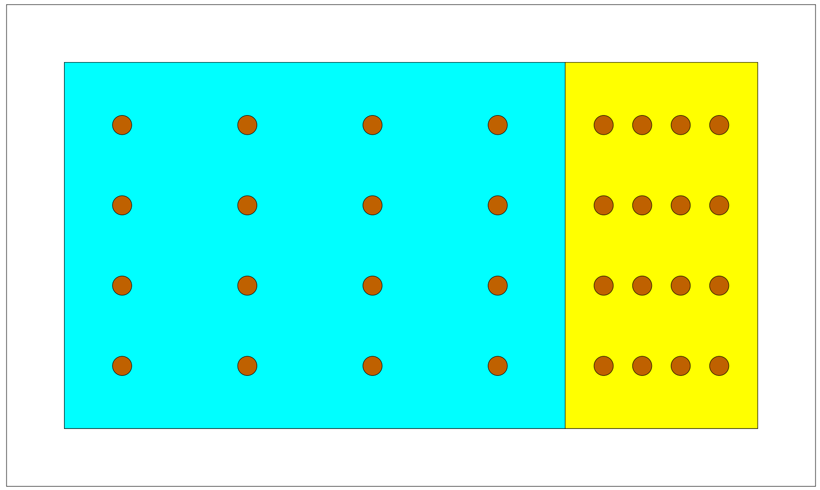


Alice and Bob share cake



A more interesting cake is shown above, with $\frac{3}{4}$ blue icing, $\frac{1}{4}$ yellow icing, and nuts distributed nonuniformly on top of the cake.

It is now possible that Alice and Bob desire *different* portions of this cake. Suppose

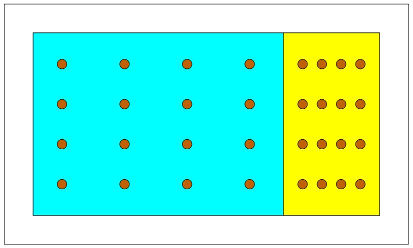
- Alice cares only for blue icing
- Bob cares only for nuts

Uncared-for aspects of the cake bring a participant no value.

Alice and Bob share cake

Alice cares only for blue icing

Bob cares only for nuts



Definition 3 proportionality*: An outcome is* proportionally fair *if each of n participants feels he or she received $\frac{1}{n}$ of the cake's value*

If Alice and Bob share the cake

- Alice must believe she receives at least $\frac{1}{2}$ of the cake's value from *her* perspective.
- It follows that she believes Bob receives no more than $\frac{1}{2}$ of the cake. Do you see why?

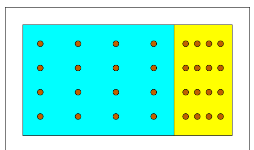
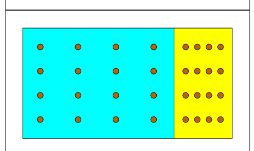
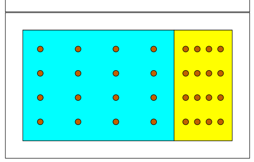
For a proportionally fair outcome, Bob must feel the same way from his perspective.

Exercise

Depending upon

- how the cake is divided, and
- who receives each piece

various results are possible. For each row below, show how the cake is divided and assigned to obtain the specified result.

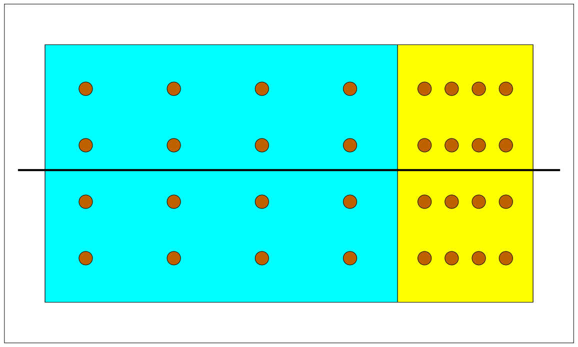
cake	Alice	Bob
	$\frac{1}{2}$	$\frac{1}{2}$
	1	$\frac{1}{2}$
	0	$\frac{1}{2}$

Exercise results

Alice cares only for blue icing, Bob cares only for nuts

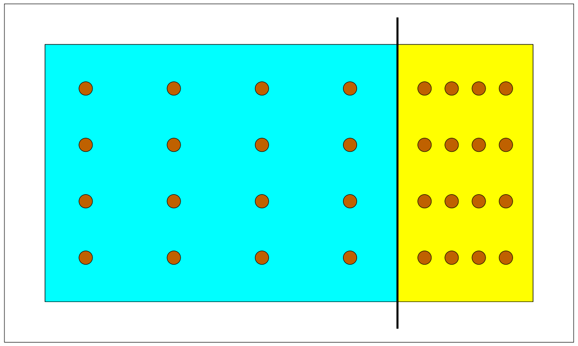
This geometric bisection of the cake yields a fair result for both Alice and Bob.

Moreover, the participants believe each received exactly $\frac{1}{2}$ of the cake's value.



Definition 4 equitable*: An outcome is* equitable *if all participants are equally happy with the result.*

Depending on how these pieces are assigned, this result is either proportionally unfair to Alice, or fair to both but not equitable.



Goals

Share cake between Bob and Alice so that the result is:

- Proportional
- Equitable
- Envy-free

  Definition 5 envy-free*: An outcome is* envy-free *if no participant prefers somebody else's assigned piece to his or her own.*

- Efficient

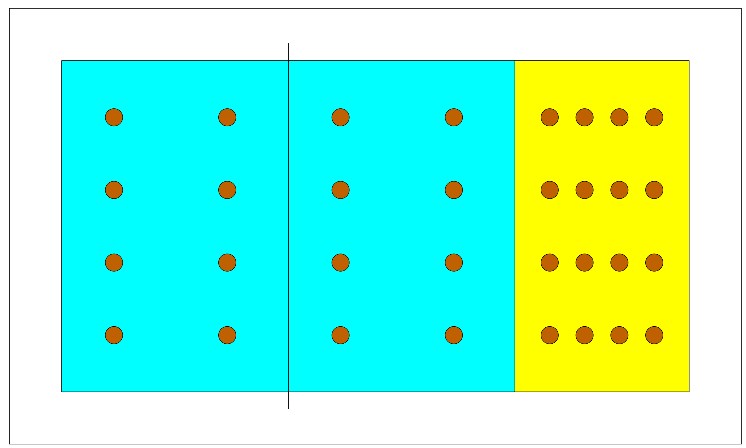
  Definition 6 efficient*: An outcome is* efficient *if none of the cake is wasted.*

Goals

Ideally, Alice and Bob *do not* diclose their preferences to each other *a priori*. Otherwise, one could cheat the other by specifying insincere preferences.

For example, suppose Bob knows Alice cares only for blue icing. If he insincerely states that blue icing is all that matters to him as well, then the division shown to the right is proportionally fair, but leaves Alice with less cake and can award Bob 50% more nuts.

If Bob receives the left piece, however, he ends up with 50% fewer nuts.



What happens if Alice says insincerely that she values only nuts?

Bram's characterization of fair divison problems

number of parties Some problems become easier for smaller numbers of participants

goods are physical objects, the items to be divided

issues are matters of differing (perhaps opposing) views

resolution framework is a mechanism (algorithm) for achieving fair division. This is broken into two steps:

1. Parties agree on a *procedure* for resolving the problem
2. The procedure is invoked and an outcome is achieved

role of preferences

- How do the parties assess their preferences for the goods?
- How do they express those preferences?
- Are the preferences open or secret?

rules are legal processes that are verifiable by a referee, without knowing preferences.

- Example: "Divide the cake into two pieces"
- Counterexample: "Divide the cake into two pieces of equal value"

procedure is a sequence of rules

> **Definition 7** impartial procedure*: A procedure is* impartial *if it favors no particular party*

strategies are actions based on procedures, rules, and private knowledge (*e.g.*, preferences)

algorithms are procedures with provable properties. These properties usually relate to the outcome, but can also include the time or space necessary for the algorithm to execute.

criteria of satisfaction What do we mean by fair?

- proportional

- envy-free
- equitable
- efficient
  - no slack (no wasted cake)
  - better still:

Definition 8 Pareto efficient solution*: No other solution is better for some party without being worse for another.*

- stability No two parties would agree to swap what they received.

  Is a stable solution necessarily envy-free?

Formalization of preferences

Given a cake K and n participants:

π	partition (cuttings) of a cake K into n pieces
π_i	piece assigned to party i
$v_i(x)$	i^{th} party's value function for x. The range of this function is $[0, 1]$

We require $\forall\ i\ v_i(\emptyset) = 0$ and $v_i(K) = 1$.

All value functions must be montonically nondecreasing. In other words, a cake has no poison: taking on more cake may bring no added value to a party, but it cannot bring "negative" value.

$v_i(S) = \sum_{x \in S} v_i(x)$. The value of a cake's pieces is the sum of the pieces' values, and no value is lost by cutting.

Restatement of satisfaction criteria

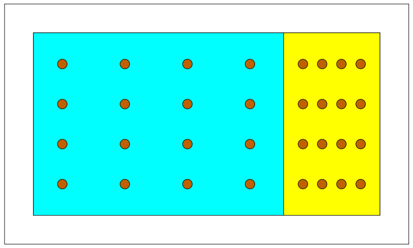
$v_i(\pi_j)$ is the value party i sees in the piece assigned to party j.

Criterion		Statement in logic
proportional	$\forall\ i$	$v_i(\pi_i) \geq \frac{1}{n}$
envy-free	$\forall\ i\ \forall\ j$	$v_i(\pi_i) \geq v_i(\pi_j)$
equitable	$\forall\ i\ \forall\ j$	$v_i(\pi_i) = v_j(\pi_j)$
stable	$\forall\ i\ \forall\ j$	$v_i(\pi_j) \leq v_i(\pi_i)$ or $v_j(\pi_i) \leq v_j(\pi_j)$
slack-free		$\cup_i\ \ \pi_i = K$

Theorem 1 *for $n = 2$, a proportional outcome is always envy-free.*

Proof: Exercise ■

How is cake usually divided between two people?



Let's assume we have just the two participants: you and some other person.

- No dictator: "Eat the piece I give you or no piece at all!"
- Parties are self-interested
- This is the only contest
- Preferences are secret
- Desire a proportional (thus envy-free) result

How would you do this?

Our first procedure: Divide and Choose (DC)

- Dates back to 700 BC, in Hesiod's *Theogony*, with meat divided between Prometheus and Zeus
- A procedure of two rules, performed in sequence:

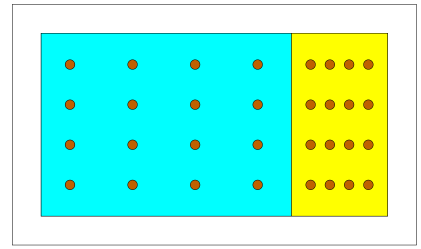
  Rule D: One player (the divider) divides the cake into two pieces

  Rule C: The other player (the chooser) chooses first

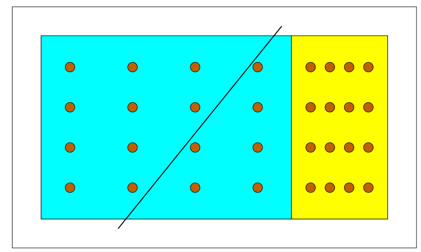
Would you rather be the divider or the chooser? Why?

Analysis of Divide and Choose: the cutter

The divider and chooser are shown the piece of cake. Each has undisclosed preferences. Without loss of generality, we assume P_1 is the cutter and P_2 is the chooser.



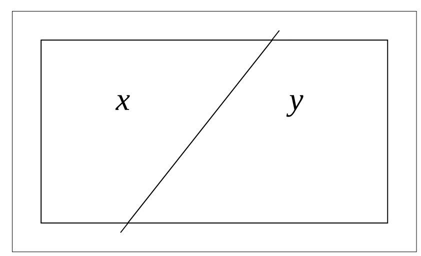
P_1 makes a single cut, which divides the cake into two pieces, according to Rule **D**.



P_1's s *strategy* must guarantee that the P_1 receives at least half of the cake's value from his or her perspective.

Because the assignment of pieces is determined by P_2, P_1's strategy must cut the cake into two regions, x and y, such that:

$$v_1(x) = v_1(y) = \frac{1}{2}$$



Analysis of Divide and Choose: the chooser

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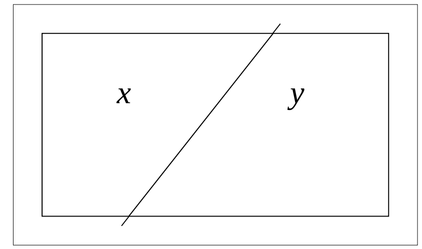
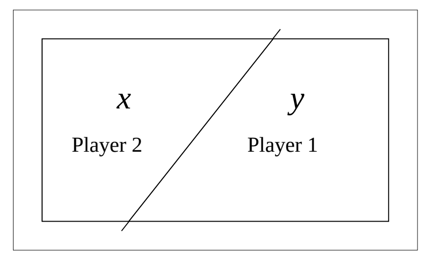


$$v_1(x) = v_1(y) = \frac{1}{2}$$

In this way, no matter how P_2 chooses, P_1 receives half the cake's value from his or her perspective.

P_2 chooses first, which effectively determines the assignment of each of the two pieces. P_2's strategy must pick the piece that has the greater value. Because $v_2(x \cup y) = 1$, one of the pieces x or y must have at least half the cake's value from P_2's perspective. Suppose the piece of greater value is x. Then



$$\begin{aligned} v_2(x) &\geq \frac{1}{2} \\ v_1(y) &= \frac{1}{2} \end{aligned}$$

which is a proportional (and thus envy-free) outcome.

Divide and Choose: satisfaction critera

The Divide and Choose procedure possesses which of the following criteria?

- stability
- equitability
- no slack
- Pareto efficiency

Divide and choose applied to *in*divisible goods

Benjamin Franklin: If you want to know the true character of a person, divide an inheritance with him.

Let's try the Divide and Choose procedure, applied to a pile of indivisible items:

divide: P_1 makes 2 piles

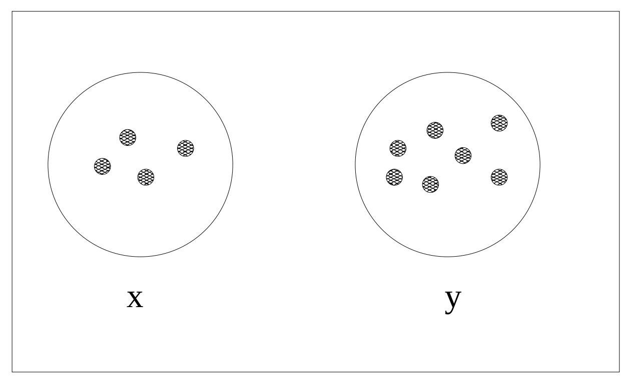
choose: P_2 chooses one pile, P_1 gets the other

But consider P_1's strategy:

P_1 must create the two piles such that

$$v_1(x) = v_1(y) = 0.5$$

But this is the PARTITION problem, which is NP-complete. Fortunately there are algorithms that can solve this for relatively small instances, using DYNAMIC PROGRAMMING.



Example from Brams[a]

Bob and Carol want to divide inherited goods. After a coin flip[b] that makes Bob the divider, he establishes two piles:

- boat, computer, moped, tractor
- boat motor, piano, moped, truck, tools, rifle

Bob really values mopeds, Carol really values the computer and piano.

- Bob believes he will get about $\frac{1}{2}$ the estate, but knows Carol might do better (why?)
- Neither side is really all that happy (Pareto-inefficient)

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[a] *Fair Division: From cake cutting to dispute resolution*, Steven J. Brams and Alan D. Taylor, 1996

[b] Why should they flip a coin to decide this?

Other examples from Brams

- James Harrington's *The Commonwealth of Oceana*, an attempt at a utopian society. Legislation is achieved by:
  - 300-person Senate proposes a bill, with lots of debate.
  - 1500-person House votes on the bill with absolutely no debate.
- US Congress, is a more symmetric version of *Oceana*. Two legislative chambers:
  - one proposes a bill
  - the other *chooses*. Here the choice is to *pass* or to *fail* the bill.

Divide and choose for indivisible items

- Not very satisfactory
- Some computationally difficult subproblems
- A better approach: ADJUSTED WINNER will be presented soon!

Can two people agree on what is exactly $\frac{1}{2}$ of a piece of cake?

- Divide and choose favors the chooser
- It is not necessarily equitable
- If both parties could agree on a piece that is exactly $\frac{1}{2}$ value
  - The other piece must also be of $\frac{1}{2}$ value
  - Because both parties must see:

$$1 - \frac{1}{2} = \frac{1}{2}$$

  - Thus, an equitable solution would be obtained
- But it is not necessarily Pareto efficient
  - Can you devise an example to show this?