# CSE 544T Notes

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**Thus far**

| | |
|---|---|
| $n = 2$ | proportional $\rightarrow$ envy free |
| $n = 3$ | Conway-Guy-Selfridge (5 [sometimes 4] cuts) |
| | Stromquist (4 knives, 2 cuts) |
| | Levmore-Cook (2 knives, 2 cuts) |
| | Webb (2 rounds, 1+2 cuts) |
| | Pie-cutting (3 knives, 1 person) |

**Algorithm**

$n = 3$ or 4, envy free, attributed to Brams

Rules ($n = 3$):

$P_1$ and $P_2$ create $X, Y, Z$

$P_3$ chooses first

$P_1$ and $P_2$ must be staisfied and not envy with any piece despite $P_3$ choosing first.

Procedure ($n = 3$):

$P_1$ and $P_2$ create $X, Y, Z$ using Austin's algorithm such that

$$\begin{aligned}
V_1(X) &= V_2(X) \\
V_1(Y) &= V_2(Y) \\
V_1(Z) &= V_2(Z) \\
\frac{1}{3}
\end{aligned}$$

$P_3$ choose first since $P_1$ and $P_2$ think all pieces are $\frac{1}{3}$.

Rules/Procedure ($n = 4$):

* Try to separate rules from procedure

$P_1$ and $P_2$ jointly create $W, X, Y, Z$, all equal to $\frac{1}{4}$.

$P_1$ and $P_2$ consider all pieces equal, so they won't envy.

$P_3$ creates a 2-way tie for largest by trimming. Without loss of generality,

$$W = W' + E \text{ via Conway-Guy-Selfridge}$$

$P_4$ chooses first. $P_3$ must choose $W'$ if available.

What about $E$?:

$P_3$ and $P_4$ $\quad$ $P_T$:
$\quad$ $P_N$:

$P_1$ or $P_2$ as normal. Irrevocable advantage over $P_T$.

$P_N$ and $P_2$ create $E_1, E_2, E_3, E_4$

$P_T$ chooses $E_i$, and is thus envy free.
$P_1$ chooses next. Doesn't envy due to irrevocable advantage over $P_T$
$P_2$ and $P_N$ pick next, in either order. Neither envies because they view all pieces equal to $\frac{1}{4}$

**Asymptotic number of cuts to cut cake**
Lemma: One Cut Suffices

Given $M$ pieces of cake, $x_1, x_2, \ldots, x_m$, it is possible to divide the cake into two equal platters, cutting only one piece to do so.

P partitions the pieces $x_i$ such that $L = V(Y_1)$ and $R = V(Y_2)$ where $L$ and $R$ are the desired value of the two platters.

1. sort $x_i$ into ascending value
2. determine $D$ such that

$$\sum_{i=1}^{D} V(x_i) \leq L < \sum_{i=1}^{D+1} V(x_i)$$

If $\sum_{i=1}^{D} V(x_i) = L$, then the one cut is not necessary. Otherwise:

$$\sum_{i=1}^{D} V(x_i) < L < \sum_{i=1}^{D} V(x_i) + V(x_{D+1}) \tag{1}$$

Let $V(x_{D+1}) = a + b$, where $a$ goes to $L$ and $b$ goes to $R$. Thus,

$$a = L - \sum_{i=1}^{D} V(x_i)$$

$$L < \sum_{i=1}^{D} V(x_i) + V(x_{D+1})$$

$$L < \sum_{i=1}^{D} V(x_i) + a + b$$

$$L < \sum_{i=1}^{D} V(x_i) + L - \sum_{i=1}^{D} V(x_i) + b$$

$$L < L + B$$

$$b > 0$$

Thus, we have shown that only one cut is required to balance $L$ and $R$ as desired.

Last diminisher (trimming):
1 cut suffices: $\frac{1}{n}$
The rest is just trimming
$(n-1) + (n-2) + \ldots + 1 = O(n^2)$ cuts

Fink's lone chooser:
Assuming gluing to make previous cuts whole:
$n = 1 \longrightarrow 0$ cuts
$n = 2 \longrightarrow 4$ cuts
$n = 3 \longrightarrow 9$ cuts

Without gluing:

$n$ people: $(n-1)(n-1)!$ cuts

But if we use 1 cut suffices:

P creates $K$ pieces of equal value and creates two platters, one with $\frac{1}{K}$ value the other with $\frac{K-1}{K}$ value. P repeats the platter division until $K$ platters which requires $K-1$ cuts, which yields $\sum_{i=1}^{K-1} i^2$ cuts which is the same as if we used glue.