Cake Cutting Algorithms

9/5/12

<u>Announcement:</u>

No class 9/17 and 9/25

<u>Today:</u>

- Formalizing Fairness
- Divide and Choose
- Moving Knife (Not started)

Some definitions:

- Rule – some activity that is verifiable by a referee.
    - e.g. cut the cake into two pieces
        - can be verified
    - not a rule to include "of equal value"
- Strategies assume self-interest (max. party's utility NOT global utility)
- Procedure – sequence of rules
- Algorithm – procedure with guarantees (of fairness or similar)

What is fair?

Criteria of satisfaction

a) Proportionality – n parties each thinks it got at least 1/n of value
    a. Literature usually refers to this as "fair"
b) Envy-free
    (remember Cain)
    No party wants another party's piece
    a. Stronger than stable marriage – pair off two groups such that no two pairs want to swap partners.
    b. Strong envy-free – every piece looks worse to me.
    c. Super envy-free – every other piece looks less than proportional.
c) Equitability – all parties think they received the same value.

* Which of these criteria imply which others for n=2? For n>2?

Back to divide and choose for 2 people

Proportional? Yes, each person gets at least half

Envy free? Yes

Cutter has ½ chooser has at least ½

Equitable? No

Stable? Yes, envy free

d) Efficiency
   a. No slack – Nothing is held back
   b. No other allocation improves one party without being worse for another. (Pareto)
e) Stability – No 2 would swap

Cake K entire cake

π partitioning – cutting of cake

π_i is piece π assigns to party i

$v_i(x)$ – i[th] party's value for piece x

range is non-negative

S = {pieces of cake}

$v_i(S) = \sum_{x \in S} v_i(x)$ "divisible" – value not diminished by division

$\forall i, v_i(k) = 1$ all parties have unit value for cake

$\forall i \forall x, v_i(x) \geq 0$ no negative utility

$v_i(\pi_i)$ i[th] party's view of π_i the piece they got.

$$\forall i \forall x \forall y, v_i(x + y) \geq v_i(x)$$

$v_i(\pi_j)$ party I's view of j's piece

a) Proportional $\forall i, v_i(\pi_i) \geq \frac{1}{n}$
   Interesting only when $\exists x \exists j \exists i, v_i(x) \neq v_j(x)$ otherwise cake looks same to everyone cut it geometrically
b) Envy-free $\forall i \forall j, v_i(\pi_i) \geq V_i(\pi_j)$
   You view your piece as at least as good as any other.

c) Equitable $\forall i \forall j, v_i(\pi_j) = v_j(\pi_j)$

d) Efficient
no slack $\bigcup_i \pi_i = 1$ whole cake is used
Pareto $\forall \pi' \neq \pi, \exists i v_i(\pi_i') > v_i(\pi_i) \rightarrow \exists j v_j(\pi_j') < v_j(\pi_j)$

Lemma 1

Assume n=2 proportional → envy-free

Proportional

$$v_1(\pi_1) \geq \frac{1}{2}$$

$$v_2(\pi_2) \geq \frac{1}{2}$$

$$v_1(\pi_1) + v_1(\pi_2) = 1$$

$$v_1(\pi_2) \leq \frac{1}{2}$$

$$v_2(\pi_1) + v_2(\pi_2) = 1$$

$$v_2(\pi_1) \leq \frac{1}{2}$$

- ➔ Envy-free

700 BC Hesiod's Theogeny

Prometheus & Zeus – fair division avoided violence

Divide and choose (DC)

Rule D: one player (1) divides the cake into 2 pieces (x and y).

Rule C: the other player (2) chooses

Self-interested, rational, secretive

P1 must therefore create portions x and y such that $v_1(x) = v_1(y) = \frac{1}{2}$.

P2 must choose x if $v_2(x) > v_2(y)$ else y

Is DC proportional?

Thm. 2 DC is proportional

$$v_1(\pi_1) = \frac{1}{2}$$

$$v_2(\pi_2) \geq \frac{1}{2}$$

Efficient? No slack but not Pareto optimal.