More on Exact Division

10/22

- Preferences File

x_1 y_1 — anything, but will scale to get unit value

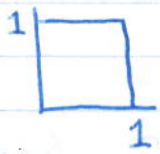
x_2 y_2

⋮

— sorted, $0 \le x_i \le 1$ strictly ascending

x	y
0	0
0	1

0	1
1	1



- Finite Algorithm
  - Exact division, 1:1

    2 parties agree on $1/2$ cake

    Moving knives works

$1/2$	$1/2$	P1
$< 1/2$	$\ge 1/2$	P2

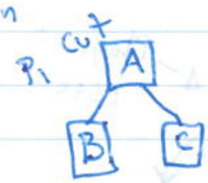
No finite algorithm can do this

within ϵ finite algorithm works (lots of little pieces)

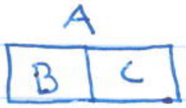
  - Exact agreement of unequal (Not 1:1) portions

No finite algorithm can accomplish exact fair division $n=2$

Intuition

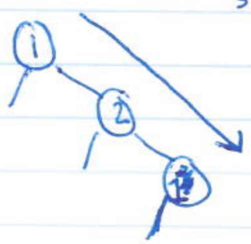


P_2 $v_2(A) = 1/3$



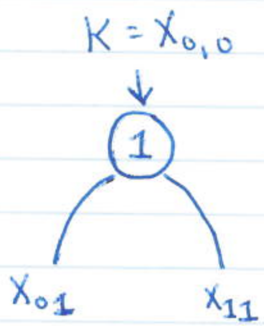
P_2 will be free to assert $\underbrace{V_2(B) + V_2(C)}_{P_2 \text{ can pick}} = V_2(A)$

P_2 can force division to continue by "special" (pathological) values for $V_2(B)$ & $V_2(C)$

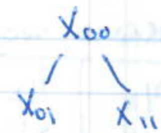


Can go on forever

<u>Step 0</u>: Cake $K = X_{0\,0}$ round 0 $X_{\text{piece, round}}$
piece 0



2 pieces from cutting

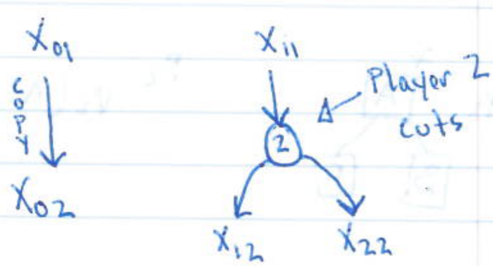


P_1 may think $v_1(X_{01}) = v_1(X_{11}) = 1/2$
(job is done)

P_2 may disagree

$V_2(X_{01}) < V_2(X_{11})$
$1/3 \qquad 2/3$

round 2
pieces 0, 1, 2



Example:

X_{00}

$V_1(X_{00}) = 1$

$V_2(X_{11}) = 1$

P_1 cuts $X_{00} \rightarrow X_{01}\ X_{11}$

	X_{00}	X_{11}
V_1	1/2	1/2
V_2	1/3	2/3

— P_2 cut this piece

	X_{02}	X_{12}	X_{22}
V_1	1/2	1/4	1/4
V_2	1/3	1/6	1/2

Both see a solution but plates are not the same

renumber

	X_{02}	X_{12}	X_{22}
V_1	1/4	1/4	1/2
V_2	1/2	1/6	1/3

— P_1 cut

	X_{03}	X_{13}	X_{23}	X_{33}
V_1	1/4	1/4	1/4	1/4
V_2	1/2	1/6	2/9	1/9

However one person cuts the rightmost piece, other player can assign value so that there is no solution

For simplicity divide the right most piece

round K $\quad x_{K,K} \rightarrow$

round K+1 $\quad x_{K,K+1} \quad x_{K+1,K+1}$

$$\begin{tikzcd}
\text{round } K & x_{0K} \arrow[d, "\text{copy}"'] & x_{1K} \arrow[d, "\text{copy}"'] & \cdots & x_{K,K} \arrow[dl] \arrow[d] \\
\text{round } K+1 & \underbrace{\qquad\qquad}_{\text{copied}} & & x_{K,K+1} & x_{K+1,K+1}
\end{tikzcd}$$

$$\forall_i \ \ i < K \quad \begin{cases} V_1(x_{i,K+1}) = V_1(x_{i,K}) \\ V_2(x_{i,K+1}) = V_2(x_{i,K}) \end{cases}$$

Preferences on copied pieces of cake can't change!

P_2 cut

$$x_{K,K} \longrightarrow \underbrace{x_{K,K+1}} \quad \underbrace{x_{K+1,K+1}}$$

Value of each to P_2, the cutter, >0

To show this can go on forever show P_1 can view new pieces such that 50/50 balance is not yet achieved

$\exists \ v_1(X_{K,K+1}) \quad v_1(_{K+1,K+1}) \mid$ exact fair division <u>Not</u> achieved

Note when P_2 cut, we weren't done

$$\rightarrow \nexists\ S \subseteq \{0, 1, \ldots, k\} \mid \sum_{i \in S} v_1(x_{ik}) = 1/2$$
$$= \sum_{i \in S} v_2(x_{ik}$$

No agreement on exact fair division

Player 1 can look at all possible subsets of
$$\{x_{0k}, x_{1k}, \ldots, x_{k-1,k}\}$$

Find a subset whose sum is closest to $1/2$ without being $1/2$

Call that subset T. Call the sum of those pieces t.

$$\eta = |1/2 - t|$$

$$v_1(x_{k,k+1}) = \frac{\min(\eta, v_1(x_{k,k}))}{2}$$

$$\begin{array}{ccc} & x_{k,k} & \\ \swarrow & & \searrow \\ x_{k,k+1} & & x_{k+1,k+1} \end{array}$$

This makes P_1 unsatisfied

Why does this "fail" to satisfy P_1?

Suppose it does satisfy P_1

$$\exists\ S \subseteq \{0, 1, \ldots, k+1\} \mid \sum_{i \in S} v_1(x_{i\,k+1}) = 1/2$$
$$= \sum_{i \notin S} v_1(x_{i,k+1})$$

impossible

- why?

P_2's point of view $\underbrace{x_{k,k+1} \quad x_{k+1,k+1}}_{\text{Not on same plate}}$

Assume $X_{K,K+1}$ is in S

$K \in S$

$K+1 \notin S$

a) if Σ without $X_{K,K+1}$ is $1/2$ then including $X_{K,K+1}$ exceeds $1/2$ $v_1(X_{K,K+1}) > 0$

b) if the Σ is not $1/2$

adding $X_{K,K+1}$ won't reach $1/2$

careful choice of $v_1(X_{K,K+1})$

Can go on forever

No finite algorithm for exact division

- Here's an algorithm that almost does the job

Pick ϵ small value tolerance for how far from $1/2$ you get

1) P_1 cuts cake A into pieces each worth $< \epsilon/2$ to P_1

2) P_2 looks at those pieces, cuts if necessary so each is worth $< \epsilon/2$ to P_2

$$A = a_1 a_2 \dots a_n \mid v_1(a_i) < \epsilon/2 \quad v_2(a_i) < \epsilon/2$$

to get pieces worth $< \epsilon/2$ take $\lceil 2/\epsilon \rceil$ cuts

[recall exam $\frac{a}{b}$ $\frac{3}{8}$

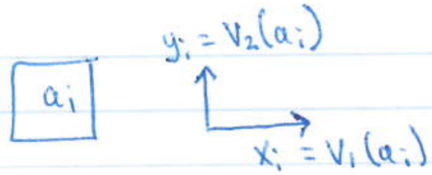
$\lfloor 8/3 \rfloor$ pieces

Do this perhaps 2y $\boxed{3/8}$ $\boxed{3/8}$ $\boxed{<3/8}$

when P_2 is done cutting

$a_1\ a_2 \dots a_m \quad m \leq 2 \lfloor 2/\epsilon \rfloor$

$\epsilon = 1/100 \quad m \leq 2(200) = 400$ pieces

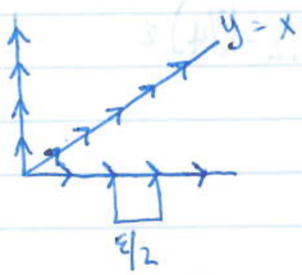


How much P_1 likes

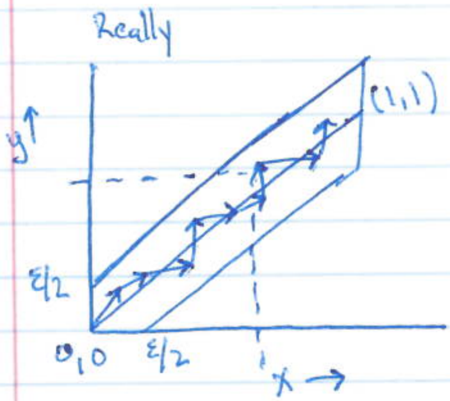
For intuition assume

$\forall_i \quad x_i = y_i$

all pieces liked the same by P_1 & P_2



← case where each piece is liked the same by each person



first choice $y > x$

1) Pick any a_1 piece to start

$x_1 < \varepsilon/2 \qquad y_1 < \varepsilon/2$

perhaps $x_1 \neq y_1$

2) Add next piece to make trajectory most closely follow $y = x$

(*) It is always possible to find such a next piece

<u>Lemma</u>: Vertical distance from any current endpoint to $y = x$ line is less than $\varepsilon/2$

(*) Proof [inductive]

each change in $x < \varepsilon/2$

worst case for finding split

Value to P_1 is within $1/2 - \frac{\varepsilon}{4}$

y is within $\varepsilon/2$ of x

P_2's view of split $1/2 - \varepsilon/4 - \varepsilon/2 = 1/2 - (3/4)\varepsilon$