1. For each statement below, determine whether the statement is true or false. Prove the statement directly from the definition if it is true, and give a counterexample if it is false. (5 points each)

a. For all integer a, b, c, if a | (b+c), then a | b or a | c.

*Counterexample: Let a = 2, b = 3, and c = 1. Then a | (b + c) because 2 | 4 but a \not| b because 2 \not| 3 and a \not| c because 2 \not| 1.*

b. For all integer a, b, and c, if a | bc, then a | b or a | c.

*Counterexample: Let a = 6, b = 2, and c = 3. Then a | bc because 6 | 6 but a \not| b and a \not| c because 6 \not| 2 and 6 \not| 3.*

c. For all integer a and b, if a | b, then a^2 | b^2.

*Proof: Let a and b be integers such that a | b. By definition of divisibility, b = ak for some integer k. Squaring both sides of this equation gives b^2 = (ak)^2 = a^2k^2. But k^2 is an integer (being a product of the integer k times itself). Hence by definition of divisibility, a^2 | b^2.*

d. For all integer a and n, if a | n and a \leq n, then a | n

*Counterexample: Let a = 4 and n = 6. Then a | n^2 and a \leq n because 4 | 36 and 4 \leq 6, but a \not| n because 4 \not| 6.*

2. Prove that for all integers n, n^2 - n + 3 is odd. (5 points)

*Proof (from definitions): Suppose n is any integer. By the quotient remainder theorem with n=2, n is either even or odd.

Case 1 (n is even): In this case n = 2k for some integer k, and so, by substitution, n^2 - n + 3 = (2k)^2 - 2k + 3 = 4k^2 - 2k + 2 + 1 = 2(2k^2 - k + 1) + 1. Let t = 2k^2 - k + 1. Then t is an integer because products, differences, and sums of integers are integers. Hence, n^2 - n + 3 = 2t + 1 where t is an integer, and so, by definition of odd, n^2 - n + 3 is odd.

Case 2 (n is odd): In this case n = 2k+1 for some integer k, and so, by substitution as above, n^2 - n + 3 = ... = 2(2k^2 +2k + 1) + 1. Let t = 2k^2 + 2k + 1. Then t is an integer because products and sums of integers are integers. Hence, n^2 - n + 3 = 2t + 1 where t is an integer, and so, by definition of odd, n^2 - n + 3 is odd.

Thus in both cases n^2 - n + 3 is odd.

3. For each of values n and d given below, find integer q and r such that: n = dq+r and 0 \leq r < d. (2 points each)

a. n = 62, d = 7

q = 8, r = 6

b. n = 3, d = 11

q = 0, r = 3

c. n = -27, d = 8

q = -4, r = 5
4. Prove that if \( n \) is any even integer, then \( \text{Floor}_\text{Of}(n/2) = n/2 \). (5 points)

Proof: Suppose \( n \) is any even integer. By definition of even, \( n = 2k \) for some integer \( k \).

Then,

\[
\text{Floor}_\text{Of}( n/2 ) = \text{Floor}_\text{Of}( 2k/2 ) = \text{Floor}_\text{Of}(k) = k \quad \text{... because } k \text{ is an integer}
\]

But,

\[
k = n/2 \quad \text{... because } n = 2k.
\]

Thus, on the one hand, \( \text{Floor}_\text{Of}( n/2 ) = k \), and on the other hand \( k = n/2 \).

It follows that \( \text{Floor}_\text{Of}( n/2 ) = n/2 \), as was to be shown.

5. Suppose that \( n \) and \( d \) are integers and \( d \neq 0 \). Prove the following. (4 points each)
a. If \( d \mid n \), then \( n = \text{Floor}_\text{Of}(n/d) \times d \)

Proof: Suppose \( n \) and \( d \) are integers with \( d \neq 0 \) and \( d \mid n \). Then \( n = d \times k \) for some integer \( k \). By substitution and algebra, \( \text{Floor}_\text{Of}(n/d) = \text{Floor}_\text{Of}(d \times k/d) = \text{Floor}_\text{Of}(k) \).

And \( \text{Floor}_\text{Of}(k) = k \), because \( k \leq k < k+1 \) and both \( k \) and \( k+1 \) are integers. But since \( n = d \times k \), then \( k = n/d \). Hence \( \text{Floor}_\text{Of}(n/d) = k = n/d \). Therefore, \( n = \text{Floor}_\text{Of}(n/d) \times d \).

b. If \( n = \text{Floor}_\text{Of}(n/d) \times d \), then \( d \mid n \)

Proof: Suppose \( n \) and \( d \) are integers with \( d \neq 0 \) and \( n = \text{Floor}_\text{Of}(n/d) \times d \). By definition of floor, \( \text{Floor}_\text{Of}(n/d) \) is an integer. Hence, \( n = d \times (\text{some integer}) \), and so by definition of divisibility, \( d \mid n \).

6. Prove for all real numbers \( x \) and \( y \), if \( x \) is irrational and \( y \) is rational, then \( x + y \) is irrational. (5 points)

Proof: Suppose not. That is, suppose there are real numbers \( x \) and \( y \) such that \( x \) is irrational, \( y \) is rational and \( x + y \) is rational. [We must derive a contradiction.] By definition of rational, \( y = a/b \) and \( x + y = c/d \) for some integers \( a, b, c, \) and \( d \) with \( b \neq 0 \) and \( d \neq 0 \). Then, by substitution, \( x + \frac{a}{b} = \frac{c}{d} \). Solve this equation for \( x \) to obtain \( x = \frac{c}{d} - \frac{a}{b} = \frac{bd - ab}{bd} = \frac{bc + ad}{bd} \). But both \( ad - bc \) and \( bd \) are integers because products and sums of integers are integers, and \( bd \neq 0 \) by the zero product property. Hence \( x \) is a ratio of integers with a nonzero denominator, and so \( x \) is rational by definition of rational. This contradicts the supposition that \( x \) is irrational. [Hence the supposition is false, and the given statement is true.]

7. Use proof by contradiction to show that for all integers \( m \), \( 7m + 4 \) is not divisible by 7. (5 points)

Proof: Suppose not. That is, suppose there is an integer \( n \) such that \( 7m + 4 \) is divisible by 7. [We must derive a contradiction.] By definition of divisibility, \( 7m + 4 = 7k \) for some integer \( k \).

Subtracting \( 7m \) from both sides gives that \( 4 = 7k - 7m = 7(k - m) \). Since \( k - m \) is an integer (being a difference of integers), 7 divides 4. But, by Example 3.3.3, this implies that \( 7 \leq 4 \), which contradicts the fact that \( 7 > 4 \). [Thus for all integers \( m \), \( 7m + 4 \) is not divisible by 7.]