CSE 441T/541T: Advanced Algorithms **Fall Semester, 2004**

Using Euclidean TSP Approx. for General TSP Approx.

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In this handout we discuss how one can use an approximation algorithm for the TSP problem when the weights satisfy the triangle inequality (TSPWTI) to develop on approximation algorithm for the general TSP problem (TSP). More broadly, the aim of this and the discussions in class is to illustrate that one can use a reduction $L_1 \leq_p L_2$ to take an approximation algorithm A for L_2 and use it to create an approximation to L_1. Sometimes, as with for Independent-Set $\leq_p$ Clique, the resulting approximaton algorithm as the same ratio as A (see practice problem 1). In other cases, such as with Vertex-Cover $\leq_p$ Clique, the resulting approximation is arbitrarily bad (see practice problem 2). The example covered in this handout shows an example where one gets a different approximatio ratio than that for A.

Let W be the largest edge weight for the TSP input G. We do need the requirement that each edge has some finite weight. We can create a TSPWTI input G as follows. For each pair $u, v \in V$, place an edge with weight $w'(u,v) = W + w(u,v)$.

First we argue that G' satisfies the triangle inequality. That is we prove that for an arbitrary three vertices x, y, z, $w'(x,y) + w'(y,z) \geq w'(x,z)$. We use a proof by contradiction. Suppose that $w'(x,y) + w'(y,z) < w'(x,z)$. Then it would follow that $w(x,y) + W + w(y,z) + W < w(x,z) + W$ which implies that $W + w(x,y) + w(y,z) < w(x,z)$. Since $w(x,y)$ and $w(y,z)$ are non-negative this would imply that $W < w(x,z)$ which contradicts that W is the maximum edge weight.

Next we argue that G and G' have the same optimal tour. Since G and G' have the same set of vertices a tour T in G is also a tour in G'. Let C_T be the cost of the tour in the TSP input G and let C'_T be the cost of the tour in the TSPWTI input G' obtained by this proposed transformation. Observe that $C'_T = C_T + nW$ since there are $n = |V|$ edges in the tour each of which is larger by an additive factor of W. Since n and W are independent of the tour T it follows that the tour which minimizes C'_T also minimizes $C_T + nW$.

Now suppose we have a c-approximation A for TSPWTI. We use it to develop an approximation algorithm for TSP. Let C_T be the cost of an optimal tour T for G. As given above, the cost of tour T in G' is $C_T + nW$. Hence, the c-approximation for TSPWTI when given G' as input will yield some tour T' with cost at most $c(C_T + nW) = c \cdot C_T + cnW$. Finally, the cost of tour T' for input G is $c \cdot C_t + cnW - nW = c \cdot C_t + (c-1)nW$ since the cost for each of the n edges is reduced by W. Thus we obtain an approximation ratio for TSP of $\frac{c \cdot C_T + (c-1)nW}{C_T} = c + \frac{(c-1)nW}{C_T}$ where C_T is the cost of the optimal tour. If you want to remove the dependence on C_T then for W_{min} the minimum edge weight the above yields an approximation ratio of $c + \frac{(c-1)nW}{nW_{min}} = c + \frac{(c-1)W}{W_{min}}$. So note that if there is a bound of r on the ratio of the maximum to minimum edge weight then this yields a $(c + (c-1)r)$-approximation algorithm for general TSP. So part of what makes it so hard to approximate is that there, in general, is no bound on the ratio between the maximum and minimum edge weight.

Finally, note that the above approximation algorithm for TSP is not a c' approximation for any constant c' since the ratio between W and W_{min} can be made arbitrarily high.