

## Homework Assignment 3

October 12, 2004

Due Date: October 26

**Core Problems**

You will receive some more problems on proving a problem is NP-complete in the next homework. One of the hard things in proving a problem is NP-complete is to decide what to reduce from. The aim of this homework is to guide you through creating reductions when you know what to reduce from. Then in the next homework you'll have to also determine what to reduce from.

For problems 3 and 4 of this homework, you need just consider doing a reduction for SUBSET-SUM. In SUBSET-SUM the input is a set of  $n$  integers  $X = \{x_1, \dots, x_n\}$  and an integer  $t$ . The question is whether or not there is subset of  $S \subseteq X$  such that  $\sum_{x \in S} x = t$ . That is, the question is whether or not sum subset of the elements in  $X$  sum to exactly  $t$ .

1. (10 pts) Give a linear program that could be used to find an optimal solution to the following problem. You do NOT need to actually solve the linear program.

The ACME Mine Company owns two different mines that produce an ore which, after being crushed, is graded into three classes: high, medium and low. The company has contracted to provide a smelting plant with 12 tons of high-grade ore, 8 tons of medium-grade ore, and 24 tons of low-grade ore per week. The two mines have different operating characteristics as detailed below:

Mine	Cost per day (\$1000)	Production (tons/day)		
		High	Medium	Low
X	180	6	3	4
Y	160	1	1	6

The goal is to determine how many days per week each mine should be operated to minimize costs will still fulfilling the smelting plant contract.

2. (15 pts) In the scheduling with deadlines (SWD) problem, you are given  $n$  events where no two events can run at the same time. Event  $i$  is specified by three non-negative integers, an earliest start time  $s_i$ , a length  $\ell_i$  and a latest allowed finishing time  $f_i$ . (Event  $i$  can be started at any time  $t$  that satisfies  $s_i \leq t$  AND  $t + \ell_i \leq f_i$ .) The question is whether or not it is possible to legally schedule all  $n$  events. In the PARTITION problem, the input is a set  $X = \{x_1, \dots, x_n\}$  of non-negative integers, and the question is whether or not there is a subset  $S \subseteq X$  such that  $\sum_{x \in S} x = \sum_{x \in X-S} x$ . PARTITION is NP-complete.

- (a) Professor P.T. Reduction believes SWD is NP-hard. To prove this he proposes the following transformation function  $T$  from a PARTITION input to a SWD input. Given  $X = \{x_1, \dots, x_n\}$ , first compute  $B = \sum_{i=1}^n x_i$ . The input for SWD is as follows. There will be  $n + 1$  events. For event  $1 \leq i \leq n$ , let  $s_i = 0$ ,  $\ell_i = x_i$  and  $f_i = B + 1$ . Finally, he includes a  $(n + 1)^{st}$  event which will serve as a divider to split the other events into two sets. For this event, set  $s_{n+1} = 0$ ,  $\ell_{n+1} = 1$  and  $f_{n+1} = B + 1$ . Does this reduction satisfy the requirements to show that SWD is NP-hard? If not, state exactly which required condition fails and prove that it fails.

- (b) Fix P.T. Reduction's reduction to prove that SWD is NP-complete.
3. (10 pts) Prove that the following multiprocessor scheduling problem, MP-SCHED is NP-hard. As input you are given a set  $A$  of jobs where for  $a \in A$  it has length  $\ell(a)$ . You are also given an integer  $m$  which is the number of processors and an integer deadline  $d$ . Each job can run on any of the  $m$  machines but only one job can run at a time on a given machine. The question is whether or not there is a partition of the jobs into  $m$  sets  $A = A_1 \cup A_2 \cup \dots \cup A_m$  where  $A_i$  is the set of jobs that will run on machine  $i$  such that for  $i = 1, \dots, m$ ,  $\sum_{a \in A_i} \ell(a) \leq d$ . In other words, is there a schedule in which all jobs are processed by time  $d$ ?
4. (25 pts) For each of the following problems either give a polynomial-time algorithm to find an optimal solution *by formulating it as a linear program* or prove that the decision version of the problem (which you should clearly state) is NP-complete.
- (a) Consider a variation of the maximum flow problem where the flow on each edge must be either 0 or the capacity of the edge. More specifically, in this problem you are given a directed weighted graph  $G = (V, E)$  with a specified source vertex  $s \in V$  and a specified sink vertex  $t \in V$ . The goal is to assign each edge  $e$  a flow value of either 0 or  $w(e)$  such that the flow into  $t$  is maximized. As in the standard maximum flow problem, for each  $v \in V - \{s, t\}$  the flow into  $v$  must equal the flow out of  $v$ .
- (b) A governmental planning agency for California wishes to determine the sources to purchase fuel for use by  $n$  depots from among  $m$  bidders. The maximum quantity offered by bidder  $i$  is  $a_i$  gallons at  $c_i$  dollars per gallon. The demand at depot  $j$  is  $b_j$  gallons. Let  $d_{ij}$  be the cost per gallon for the delivery from bidder  $i$  to depot  $j$ . The goal is to meet the demands at all  $n$  depots with the minimum possible total cost (fuel and shipping).

### Advanced Problem (Required only for CS 541T students)

5. (10 pts) Prove that there is a polynomial time algorithm  $A$  that solves SAT (i.e. given a boolean formula  $\phi$ ,  $A$  respond "yes" if and only if  $\phi$  has a satisfying assignment) if and only if there is some polynomial time algorithm  $B$  that when given a boolean formula  $\phi$  it computes a satisfying assignment for  $\phi$  (or reports none exists).