CS 441T/541T: Advanced Algorithms **Fall Semester, 2004**

Homework Assignment 0

September 2, 2004 *Due Date: None*

Based on past experience, the most important prerequisite for this course is the ability to create and write proofs and to realize when you have failed to properly complete a proof. The proof techniques that are used most in this course are proof by contradiction (or indirect proof), induction, and proof by cases.

To help each of you evaluate your background in using these proof techniques, I am providing this *optional* homework. This homework will in no way be part of your grade and thus it would be pointless to look up solutions in notes or books that you have. Also, remember it is much easier to understand a proof than create one and so this will be of little value if a friend tells you how he/she did the proof – even if you understand it completely once it is explained to you. The goal here is to see if you can solve these problems on your own. Some of these proofs are non-trivial but so are the proofs you will be doing throughout this course. If you need some guidance that is fine. Just come by my office hours or make an appointment. If you aren't sure about whether or not your proofs for each of these problems are valid proofs, then I recommend that you submit your solutions. I will read them and put comments on for you and return them. I will try to return it one class period after I receive it.

1. Let $P(n)$ be that any n lines, where no two are parallel and no three pass through the same point, divide the plane into n^2+1 regions. What is wrong with the following inductive proof? It is not sufficient to give a counterexample to the given theorem. Rather, you must find the flaw in the proof.

   Theorem: $\forall n \geq 1,\ P(n)$

   Proof: By mathematical induction on n.
   Basis Step: 1 line divides the plane into 2 regions and $1^2+1=2$. Hence $P(1)$ is true.

   Inductive Step: We must show that $\forall n \geq 1\ P(n) \rightarrow P(n+1)$. By the inductive hypothesis there are n^2+1 regions formed with n lines. Note that $(n+1)^2+1 = n^2+1+2n+1$. So adding the $(n+1)$st line creates $2n+1$ new regions. Hence the number of regions with $n+1$ lines is $n^2+1+2n+1 = (n+1)^2+1$

   Since $P(1)$ is true and $\forall n \geq 1,\ (P(n) \rightarrow P(n+1))$, by the principle of mathematical induction we have that $\forall n \geq 1,\ P(n)$. ■(end of proof)

2. Prove that n lines separate the plane into $(n^2+n+2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

3. Let $P(n)$ be the predicate that there are n^2 two scoop options with n flavors where both scoops <u>can</u> be the same flavor. Note that flavor 1 on top of flavor 2 is to be viewed as a different option than flavor 2 on top of flavor 1. Prove (using mathematical induction) that $\forall n \geq 1,\quad P(n)$.

4. A *perfect number* is an integer which is equal to the sum of all its divisors except the number itself. Thus 6 is a perfect number, since $6 = 1+2+3$, and so is 28. By definition, 1 is not considered to be a prime number.

   Prove that a perfect number is not prime.

5. Prove that there is no largest prime number. That is, prove that for any prime number p, there is a prime number p' such that $p' > p$.

6. Suppose you have a set of homes that you want to connect via a communications network. Assume that you can directly place a connection between any two homes with a cost that is proportional to the distance between the two homes. Your goal is to choose which set of connections to install so that you incur the minimum possible cost under the constraint that you must ensure that all homes are connected (i.e. between any two homes you can follow a sequence of connections from one to the other)

   Prove that there exist a solution to this problem that is optimal (i.e. it connects all the homes and has the minimum possible cost) which puts a direct connection between the two closest homes.

7. Prove that for all configurations of four points on a piece of paper, there exists a way to color the points (where each point must be colored red or blue) such that there is *no* line for which all the blue points are on one side of the line and all the red points are on the other side of the line.

   Hint: Use a proof by cases.