CSE 441T/541T Advanced Algorithms **Fall Semester, 2004**

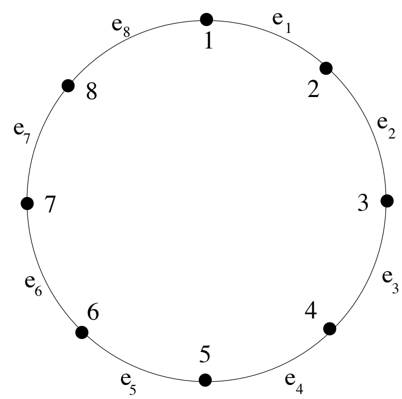
Exam 2

November 16, 2004

NOTE: For problems 1 and 2 you will have the option of solving a different (easier) problem for a lower maximum score. We will NOT allow you to solve both and take the better score. If you submit an alternate problem (without clearly telling us to ignore it) then that is the problem we will grade.

All of the problems in this exam will look at two variations of the Ring Loading Problem. I expect you to spend 10-15 minutes acquainting yourself with this problem. I've included a input and solution to help you with this process.

In the Ring Loading problem you are given an n-node network in a ring topology. Below is such a topology for $n = 8$. We use e_i to denote the edge going between node i and node $i+1$ for $i = 1, \ldots n-1$, and e_n is the edge going between node n and node 1.



You are given a set of m connections that must be set up where connection k (for $k = 1, \ldots, m$) is specified by a bandwidth demand d_k, a source node s_k, and a destination node t_k where d_k is any non-negative integer and s_k and t_k are each integers between 1 and n. Since the network has undirected edges, the source and destination for each connection are interchangeable. Thus, for simplicity, we assume that $s_k < t_k$ for each connection k.

Note that a pair of nodes can have multiple connections that involve them. For each of the m connections there is a choice to route it by going clockwise or counterclockwise. Let the *bottleneck bandwidth* B for the network be the demand routed along the link that carries the most bandwidth among all n links in the ring. The goal is to minimize the bottleneck bandwidth B. We look at the following two problems here:

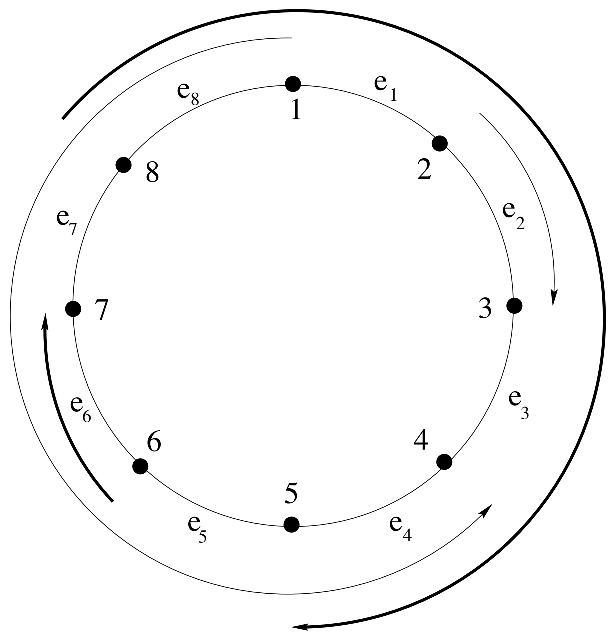
Splittable Ring Loading Problem (S-RLP). Here you can route any fraction of each connection clockwise and the rest counterclockwise. For example, if the demand for connection k were 2. You could route 1.2 units of connection k clockwise and 0.8 units of connection k counterclockwise. The amount routed in each direction for each connection can be independently selected.

Non-Splittable Ring Loading Problem (NS-RLP). Here you must route all the traffic for a given connection in the clockwise or counterclockwise direction. Different connections (even if they involve the same pair of nodes) can go in different directions.

Let's look at an example to be sure the problem definition is clear. Here is an example where $n = 8$ and $m = 4$.

k	d_k	s_k	t_k
1	1	2	3
2	1	1	4
3	2	5	8
4	2	6	7

An optimal solution to NS-RLP for this input is obtained by routing connections 1,3, and 4 clockwise and connection 2 counterclockwise. This achieves a bottleneck bandwidth of 3. To help visualize the solution, the connections are shown below where a thin line represents a connection of 1 and a thick line represents a connection of 2.



Note that all edges with an even index have 3 units of bandwidth along them. The edges with an odd index have either 1 or 2 units of connection along them.

For this particular input, it is not hard to see that this solution is also optimal for the S-RLP version where one could send any fraction of a connection clockwise and the rest counterclockwise.

1. (35 pts) Give a polynomial time algorithm for S-RLP using linear programming. Explain why your LP formulation is correct and argue that the resulting solution gives a polynomial time algorithm for S-RLP. For a 5 point hint we will give you the variables to use. For an additional 5 point we will write the optimization function. You also have the option of solving a much easier LP problem for a maximum score of 15 points. I will give you that problem upon request.

   The following notation may be helpful to you.

$$\begin{aligned} C(e_i) &= \{k \mid \text{edge } e_i \text{ is included on the clockwise connection between } s_k \text{ and } t_k\} \\ &= \{k \mid s_k \leq i < t_k\} \end{aligned}$$

   and

$$\begin{aligned} CC(e_i) &= \{k \mid \text{edge } e_i \text{ is included on the counterclockwise connection between } s_k \text{ and } t_k\} \\ &= \{k \mid i < s_k \text{ or } t_k \leq i\}. \end{aligned}$$

   Thus for a given edge e, each of the m connections are placed in exactly one of $C(e)$ or $CC(e)$. For the example given earlier, $C(e_3) = \{2, 3\}$ and $CC(e_3) = \{1, 4\}$.

2. (35 pts) In this problem you are to prove NS-RLP is NP-complete. You are welcome to reduce from any of: CIRCUIT-SAT, FORMULA-SAT, 3-CNF-SAT, CLIQUE, VERTEX-COVER, SUBSET-SUM, PARTITION, HAM-CYCLE, TSP. If you'd ask me, I'd recommend that you pick PARTITION.

   If you cannot solve this problem, then I will give you two alternate (and I think easier) problems to choose between. Your maximum score if you switch to an alternate problem is 25 points.

   (a) (3 pts) Define a decision version of NS-RLP. In the below, I will now use NS-RLP to refer to the decision version of this problem.

   (b) (2 pts) Prove NS-RLP is in NP.

   (c) (20 pts) Give the transformation function f used in your reduction to prove that NS-RLP is NP-hard. I'll use "L" to name the problem you reduced from.

(d) (4 pts) Prove that $x \in L \rightarrow f(x) \in$ NS-RLP.

(e) (6 pts) Prove that $f(x) \in$ NS-RLP $\rightarrow x \in L$

3. (30 pts) This problem builds upon problem 1. If you did the alternate for problem 1 then I will write the solution for problem 1 for you so that you can proceed on this one. If you are uncertain about your solution to problem 1 then you can come up and ask me to grade it. The first time you ask me to do this, if there are any errors, I will let you know if they are major or minor and give you the option of trying to correct them. The second time, your problem 1 will be graded regardless of if there are problems. If your solution had errors then I will make the needed corrections so that you have the correct answer to problem 1 in front of you.

   In this problem you will design a 2-approximation for NS-RLP.

   (a) (15 pts) Give an approximation algorithm for NS-RLP using LP relaxation. Your algorithm should be given as pseudo-code or as a list of explicit steps that are to be solved. A reader who has never heard of LP relaxation should be able to understand your algorithm, so be explicit. You can assume the reader knows what a linear program is and that there is a provided polynomial time algorithm to solve a linear program.

(b) (1 pts) Argue that the approximate solution computed by your approximation algorithm is legal in that for $k = 1, \ldots, m$, connection k is routed entirely clockwise or entirely counterclockwise.

(c) (14 pts) Prove that the bottleneck bandwidth of your approximation solution is at most twice that of the optimal bottleneck bandwidth. Be explicit here and really prove your claim versus just giving the intuition. This proof is not hard which is why I selected it. My goal here is to confirm you can really prove the approximation bound versus just giving intuition without proof. I would expect to see a few summations and a minimal amount of algebra in your solution. Of course, giving only intuition is better than giving nothing, but don't expect full credit if that's what you do.

Additional space. If you use this indicate which problem it is for.

Problem	Points Possible	Points Received
1	35	
2	35	
3	30	
total	100	

Alternate Problem 1. (Max Score: 15 points)

Inter-trade company buys textiles from China, India and the Philippines, ships to either Hong Kong or Tawain for packaging and labeling, and then ships to the U.S. or France for sale. The transportation costs between sources and destinations can be read from the following table:

	China	India	USA	France
Hong Kong	\$50/ton	\$90/ton	\$150/ton	\$180/ton
Taiwan	\$60/ton	\$95/ton	\$130/ton	\$200/ton

Suppose that Inter-trade has available for shipping 60 tons of textile from China and 55 tons from India. The U.S. market demands 70 tons of labeled products and the French market 35 tons. Assume that packaging and labeling do not change the weight of the textile products. Hong Kong can process at most 50 tons, while Tawain has unlimited packing and labeling capacity. Formulate a linear program to minimize the shipping cost while meeting the demand.

Formulate this as a linear program.

Alternate Problem 2. (Max Score 25 points).

Pick ONE of the following and prove it is NP-complete. You have the option of reducing from the same set of problems given in problem 2.

- You are given an undirected graph $G = (V, E)$, a subset of its edges $E' \subseteq E$, and an integer k. The problem is to determine if G contains a cycle of length at most k that includes every edge in E'.

  I'd recommend a reduction from HAM-CYCLE.

- Here we consider the Safe-Truckers-Problem. There are n loads to be transported where load i would provide profit p_i. However, due to safety factors there are a limited number of legal combinations of objects that can be placed in a truck. There are m legal loads $L_1, \ldots, L_m$ where $L_j \subseteq \{1, 2, \ldots, n\}$ indicates which items are in load j. The goal is to maximize the profit that can be achieved with a given number of trucks where each truck can hold any one load and no two loads can be combined into one truck. Note that any item that can be safely placed in more than one of the trucks is arbitrarily placed in just one.

  The decision problem (STP) is formally stated as follows. The input are $p_1, \ldots, p_n$ (positive integers), $L_1, \ldots, L_m$ (subsets of $\{1, \ldots, n\}$), t (a positive integer representing the number of trucks), and P (a positive integer representing the desired profit). The question is whether or not there are t sets $L_{i_1}, \ldots, L_{i_t}$ such that for $L = L_{i_1} \cup \cdots \cup L_{i_t}$, $\sum_{i \in L} p_i \geq P$.

  Here's an example input to be sure the problem is clear. Let $t = 3, n = 8, m = 6, P = 30$. Let $p_1 = 1, p_2 = 2, p_3 = 3, p_4 = 4, p_5 = 5, p_6 = 6, p_7 = 7, p_8 = 8$, and $L_1 = \{1, 2, 3\}$, $L_2 = \{2, 4\}$, $L_3 = \{5, 7, 8\}$, $L_4 = \{3, 6\}$, $L_5 = \{1, 8\}, L_6 = \{2, 5, 7\}$. The answer here is "yes', since if the trucker picked the loads L_3, L_4, L_6 then the items that would be shipped are $2, 3, 5, 6, 7, 8$ with a profit of 31 (which is ≥ 30).

  I'd recommend a reduction from VERTEX-COVER.