Expected Chain Length

Note Title 2/23/2008

X = length of chain hash table[slot] in separate chaining where m is hash table size + there are n elements/tags

$$E[X] = \sum_{\substack{i \text{ that } X \\ \text{can take}}} i \cdot \text{Prob}[X = i]$$

$$E[X] = \sum_{i=1}^{n} i \cdot \text{Prob}\left[\begin{array}{l}\text{exactly } i \text{ elements} \\ \text{hash to given slot}\end{array}\right]$$

$$= \sum_{i=1}^{n} i \binom{n}{i} \left(\frac{1}{m}\right)^{i} \left(1 - \frac{1}{m}\right)^{n-i}$$

$$= \sum_{i=1}^{n} \frac{\cancel{i}\, n (n-1)!}{\cancel{i}\,(i-1)!\,(n-i)!} \left(\frac{1}{m}\right)^{i} \left(1 - \frac{1}{m}\right)^{n-i}$$

$$\binom{n-1}{i-1}$$

Binomial Theorem

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$E[X] = n \sum_{i=1}^{n} \binom{n-1}{i-1} \left(\frac{1}{m}\right)^{i} \left(1-\frac{1}{m}\right)^{n-i}$$

$$= n \sum_{i=0}^{n-1} \binom{n-1}{i} \Big(\underbrace{\frac{1}{m}}_{a}\Big)^{i+1} \Big(\underbrace{1-\frac{1}{m}}_{b}\Big)^{n-(i+1)}$$

$$= \frac{n}{m} \underbrace{\sum_{i=0}^{n-1} \binom{n-1}{i} \left(\frac{1}{m}\right)^{i} \left(1-\frac{1}{m}\right)^{n-1-i}}$$

$$= \frac{n}{m} \Big(\underbrace{\frac{1}{m} + 1 - \frac{1}{m}}_{=1}\Big)^{n-1} = \boxed{\frac{n}{m}}$$

$$(a+b)^{n-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} a^{i} b^{n-1-i}$$

X_j = # times that j^{th} element goes to given slot

$$\underbrace{\qquad\qquad\qquad\qquad}_{0 \text{ or } 1}$$

indicator random var

$$E[X_j] = 1 \cdot \Pr[X_j = 1] + \cancel{0 \cdot \Pr[X_j = 0]}$$

$$= \Pr[X_j = 1]$$

$$= \frac{1}{m}$$

Linearity of Expectations

$$E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n]$$

Insert n elements, X_j is random variable corresponding to j^{th} element

$$E[X] = E[\underbrace{X_1 + \cdots + X_n}_{\substack{0 \text{ if not to given slot} \\ 1 \text{ if hash to given slot}}}] = E[X_1] + \cdots + E[X_n] = \frac{n}{m}$$

\# elements/tags hash to given slot

all $\frac{1}{m}$