Note Title 1/29/2008

Master Method

Solve asymptotically any recurrence of form

$$T(n) = aT(n/b) + \Theta(n^l (\log n)^k)$$

termination $T(c) = \Theta(1)$ for some constant c

constants $a \geq 1,\ b > 1,\ l \geq 0,\ k \geq 0$

Can you apply the master method to?

$\Theta(n^\ell (\log n)^k)$

$T(1) = \Theta(1)$

$n>1 \quad T(n) = 4T(n/2) + 10n + \frac{n \log_2 n}{2} + 5n\sqrt{n}$

$a=4 \qquad b=2$

$\Theta(n^{3/2}) \qquad n^{3/2}$

$\log_b a = 2$

$\ell = 3/2, \; k=0$

$T(n) = \Theta(n^2)$

Does $T(n) = 2T(n/2) + \Theta(n/\log n)$ fit into master method?

No $\quad a=2, \quad b=2$

$l=1 \quad \cancel{k=-1}$ not allowed

Does $T(n) = 2T(n/2) + \Theta(n/\log n)$ fit into master method?

No since we would need to set k to -1 & there's a restriction that $k \geq 0$.

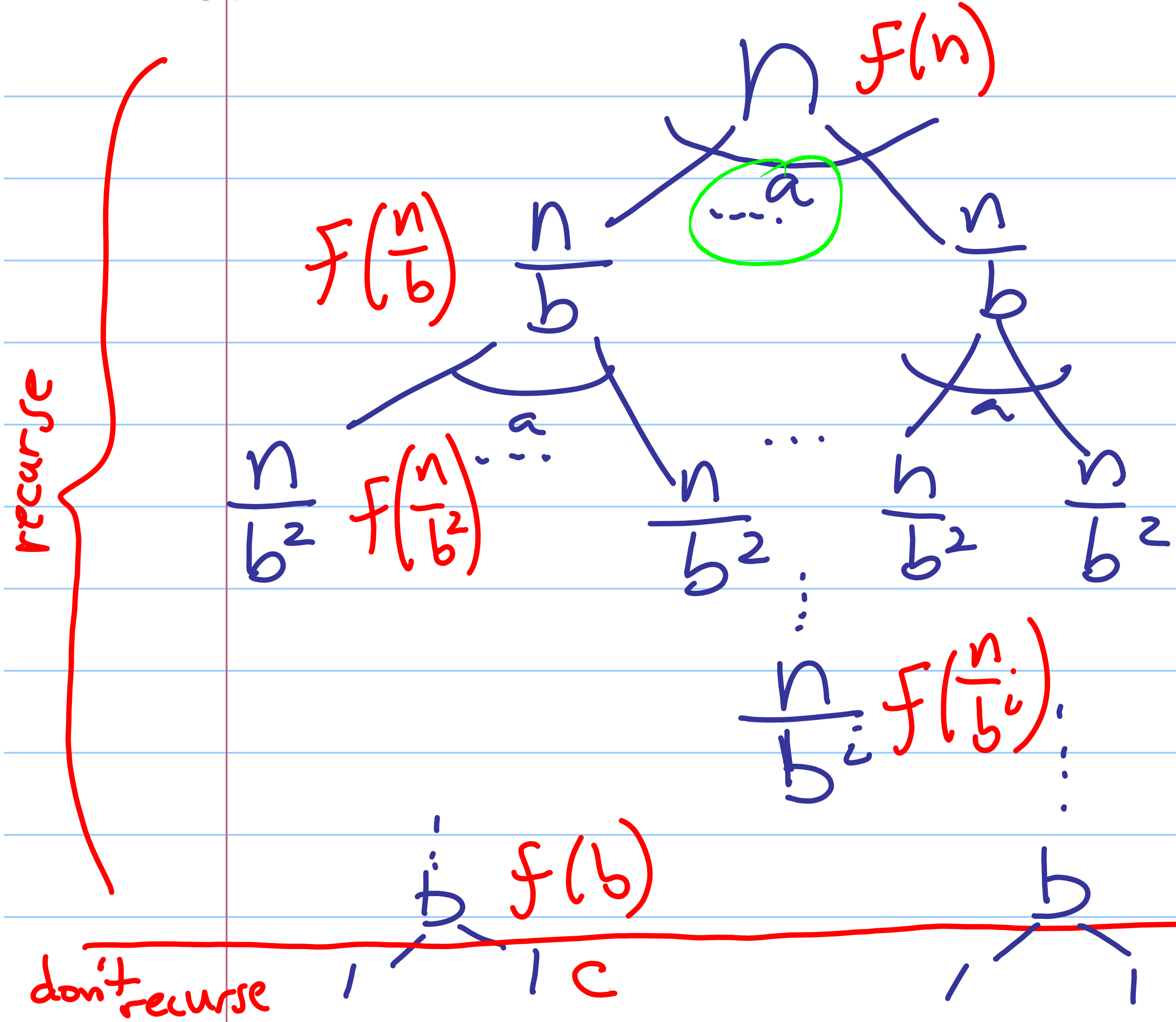
How can you exactly solve a recurrence of the form

$T(1) = \underline{c}$

$T(n) = aT(n/b) + f(n)$

where n is a power b

Recurrence Tree



level	#	time each
0	$a^0=1$	$\times\ f(n)$
1	a	$\times\ f\left(\frac{n}{b}\right)$
2	a^2	$\times\ f\left(\frac{n}{b^2}\right)$
i	a^i	$\times\ f\left(\frac{n}{b^i}\right)$
$(\log_b n)-1$		$f(b)$
$\log_b n$	$+\ a^{\log_b n}$	$\times\ c$

level #	# nodes		time per node
0	$a^0 = 1$	$\times$	$f(n)$
1	$a^1 = a$	$\times$	$f(n/b)$
2	a^2	$\times$	$f(n/b^2)$
$\vdots$	$\vdots$	$\vdots$	
i	a^i	$\times$	$f(n/b^i)$
$\vdots$	$\vdots$		$\vdots$
$(\log_b n) - 1$	$a^{(\log_b n)-1}$		
$\log_b n$	$a^{\log_b n}$	$\times$	c

recurse

$$a^{\log_b n} = \boxed{n^{\log_b a}}$$

verify by taking $\log_b$ of both sides

$$T(n) = \left[\sum_{i=0}^{(\log_b n)-1} a^i \times f(n/b^i)\right] + n^{\log_b a} \cdot T(1)$$

Master Method $f(n) = \Theta\left(n^{\ell} (\log n)^k\right)$

$\ell \geq 0,\ k \geq 0$

$$T(n) = \left[\sum_{i=0}^{(\log_b n)-1} a^i \, \Theta\left(\left(\frac{n}{b^i}\right)^{\ell} \cdot \left(\log \frac{n}{b^i}\right)^k\right)\right] + n^{\log_b a} \cdot T(1)$$

Let's Solve

$T(1) = 1, \quad T(n) = 4T(n/2) + cn$

Either find asymptotic solution with master method, or

exactly solve for n a power of 2.

geometric sum

(useful if you are going to exactly solve)

$$\sum_{i=0}^{m} x^i = \frac{x^{m+1} - 1}{x - 1} \qquad x \neq 1$$

$T(1)=1, \quad T(n)=4T(n/2)+cn$

Using master method

$$T(n)=4T(n/2)+\Theta(n)$$

$a=4, \quad b=2, \quad \log_2 4=2$

$l=1, \quad k=0$

$\log_b a > l \Rightarrow T(n)=\Theta(n^2)$

Recurrence

$$T(1) = 1, \quad T(n) = 4T(n/2) + \underbrace{cn}_{f(n)}$$

assume n is a power of 2
(b in general)

From earlier derivation

$$T(n) = \left[\sum_{i=0}^{(\log_b n)-1} a^i \times \underbrace{f(n/b^i)}_{\frac{c \cdot n}{2^i}} \right] + n^{\log_b a} \cdot \underbrace{T(1)}_{1}$$

$$T(n) = \left[\sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i)\right] + n^{\log_b a} \cdot T(1)$$

$$= \left[\sum_{i=0}^{(\log_2 n)-1} 4^i \frac{c \cdot n}{2^i}\right] + n^2$$

$$= \left(cn \sum_{i=0}^{(\log_2 n)-1} 2^i\right) + n^2 = cn\left(\underbrace{2^{\log_2 n}}_{=n} - 1\right) + n^2$$

$$= cn(n-1) + n^2 = \boxed{(c+1)n^2 - cn}$$

$$\boxed{T(n) = 4T(n/2) + cn}$$

$$\boxed{(c+1)n^2 - cn}$$

$T(1) = 1$ $\qquad$ $c+1-c=1$ ✓

$T(2) = 4T(1) + 2c$ $\qquad$ $4+2c$ $\qquad$ $(c+1)4 - 2c$

$= 2c+4$ ✓

$T(4) = 4T(2) + 4c$

$\vdots$

Sanity check

$$T(1) = 1$$

$$T(n) = T(n-1) + 5n$$

$$\sum_{i=1}^{x} i = \frac{x(x+1)}{2}$$

$$\frac{x}{2} \text{ pairs} \left(\overbrace{1 + \underbrace{2 + \cdots + (x-1)}_{x+1} + x}^{x+1} \right)$$

$T(1)=1, \quad T(n) = T(n-1) + 5n$

\# statements executed excluding recursive call

problem size	
n	$5n$
$n-1$	$5(n-1)$
$n-2$	$5(n-2)$
$\vdots$	
2	$5 \cdot 2$
1	$T(1)=1$

$$5n + 5(n-1) + \cdots + 5\cdot 2 + \overbrace{5\cdot 1 - 4}^{1}$$

$$\underbrace{5\left(\underbrace{n + (n-1) + \cdots + 2 + 1}_{\frac{n(n+1)}{2}}\right)}$$

$$\frac{5}{2} n(n+1) - 4 = \Theta(n^2)$$