

Practice Problems on NP-Completeness

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Below are two practice problems on proving that problems are NP-complete. For those of you who feel like you need us to guide you through some additional problems (that you first try to solve on your own), these problems will serve that purpose.

The front page has the problems and the rest gives the solutions. You can use these solutions as a guide as to how you should write-up your solutions. These problems will be MUCH more valuable to you if you first solve them and then check the solutions.

Practice Problems

1. Prove that the following problem, the Non-bored Jogger's Problem (NBJP), is NP-complete.

You are given as input a weighted undirected graph G (loops and multiple edges are allowed and the weights are positive integers), a specified node v , that is called home, and an integer $\ell \geq 0$. You must determine if there exists a route for the jogger (i.e. a path in G) that starts at vertex v , never repeats an edge, and returns to v after travelling a distance of exactly ℓ .

Big Hint: Reduce from SUBSET-SUM. Notice that you can create any graph you want. For example, you could choose to have a graph with only a single vertex.

2. The **subgraph-isomorphism problem** takes two undirected graphs G_1 and G_2 and asks whether G_1 is a subgraph of G_2 . In other words, the problem asks whether there is a one-to-one function f to map the vertices of G_1 to the vertices of G_2 such that there is an edge $\{u, v\}$ in G_1 exactly when $\{f(u), f(v)\}$ is in G_2 .

Show that the subgraph-isomorphism problem is **NP-Complete**.

Solutions (solve the problems before reading this)

1. Clearly NBJP is in NP since we can use the route as a certificate, and easily verify its correctness in polynomial time. Now we prove $\text{SUBSET-SUM} \leq_p \text{NBJP}$.

Let $\langle S = \{a_1, \dots, a_n\}, t \rangle$ be the SUBSET-SUM input. We construct the following NBJP input. The graph G will contain the single home vertex v with n self-loops, one associated with each element of S . The weight on the edge associated with a_i will be a_i . Finally we let $\ell = t$. Clearly this can be constructed in polynomial time.

We now argue that G contains a route that starts at v , never repeats an edge, and returns to v with distance exactly ℓ if and only if there is a solution to the subset sum problem. First note that if $S' \subseteq S$ is a solution to the subset sum problem then the jogger can obtain a solution to the NBJP by just taking those edges that correspond to the elements of S' . Likewise, if there is a solution to NBJP then the subset of S that correspond to the edges used in the NBJP solution forms a solution to the subset sum problem.

2. As the certificate use the one-to-one function f from the vertices of G_1 to the vertices of G_2 . Thus the length of the certificate is $O(n)$. Finally, given the certificate the verification algorithm can confirm that f is a one-to-one function and then take each edge $(u, v) \in G_1$ and verify that $(f(u), f(v)) \in G_2$. Clearly this can be done in $O(n+m)$ time. We now prove that $\text{CLIQUE} \leq_p \text{subgraph-isomorphism}$.

Let $\langle G, k \rangle$ be the input for clique. For the subgraph-isomorphism input we let G_1 be a complete graph on k vertices and we let $G_2 = G$. Clearly this can be done in polynomial time. To see this notice that we can assume that $k \leq n$ (or otherwise, clearly G does not have a clique of size k) and thus the time to create G_1 is simply $O(k^2) = O(n^2)$ which is polynomial in the number of bits to represent G .

By definition of a clique being a complete graph on k vertices, there is a clique of size k in G if and only if G_1 is a subgraph of G_2 . That is if there is a clique of k vertices in G then mapping the vertices of G_1 to those vertices in G_2 gives a solution for the subgraph isomorphism problem. Similarly, if there is a solution of the subgraph isomorphism problem, then the vertices in G_2 mapped to by the vertices in G_1 form a clique of k vertices.