CS 441T/539T: Advanced Algorithms	**Fall Semester, 2001**
Homework Assignment 3	
October 4, 2001	*Due Date: October 18*

Core Problems

1. (5 pts) Formulate the following as a linear program. A refinery produces two grades (A and B) of gasolines from two different sources of crude oil (I and II). Any crude oil can be used to produce any of the gasolines as long as the following specifications are met:

Grade of gasoline	Specifications	Selling price/gallon
A	at least 50% crude 1 at most 30% crude 2	$ 1.39
B	At least 35 % crude 1 at most 45 % crude 2	$ 1.24

   There are 10,000 gallons of crude 1 available at $1.10/gallon, and 9,000 gallons of crude 2 available at $.84/gallon. The oil refinery wants to maximize profit.

2. (10 pts) In SUBSET-SUM the input is a set $S = \{x_1, x_2, \ldots, x_n\}$ and the integer t. In the COMPOSITE problem you are given as input an integer y and the question to answer is whether or not y has a factor besides 1 and itself. SUBSET-SUM is NP-complete and COMPOSITE is in NP. Clearly explain whether or not each of the following statements follow from these two facts.

   (a) COMPOSITE $\leq_p$ SUBSET-SUM.

   (b) SUBSET-SUM $\leq_p$ COMPOSITE.

   (c) If there is an $O(n\sqrt{t})$ algorithm for SUBSET-SUM, then P = NP.

   (d) If there is an $O(n^3 \log t)$ algorithm for SUBSET-SUM, then there is a polynomial time algorithm for COMPOSITE.

   (e) If there is an $O(\log y)$ algorithm for COMPOSITE, then P = NP.

3. (10 pts) Either prove the following problem is NP-complete or prove it is solvable in polynomial time *by formulating it as a linear program.* There are n cities where city i produces c_i tons of textiles per day. There are m destinations where destination j must receive at least d_j tons of textiles per day. The textiles are shipped by one of p shipping companies where shipper k can transport at most s_k tons per day. Finally, for every $1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq p$, let $f_{i,j,k}$ be the fee (per ton) for shipper k to ship textiles from city i to destination j.

   Your goal is to determine if there is a way to route the textiles from the source cities to the destinations with cost at most C where all constraints stated above are satisfied.

4. (15 pts) Given a set of m linear constraints over n variables, the **integer-programming problem** asks whether there is an *integer* n-vector x giving values for each of the n variables such that all the constraints are satisfied. Prove that integer programming is NP-complete by using a reduction from 3-CNF-SAT.

   Hint: The arithmetic expression $1 - x$ performs a negation when x is 0 or 1.

5. (20 pts) In this problem we consider SCHEDULING WITH DEADLINES (SWD). Here is the problem. You are given n events where no two events can run at the same time. Event i is specified by three non-negative integers, an earliest start time s_i, a length ℓ_i and a latest allowed finishing time f_i. (Event i can be started at any time t that satisfies $s_i \leq t$ AND $t + \ell_i \leq f_i$.) The question is whether or not it is possible to legally schedule all n events.

   In the PARTITION problem, the input is a set $X = \{x_1, \ldots, x_n\}$ of non-negative integers, and the question is whether or not there is a subset $S \subseteq X$ such that $\sum_{x \in S} x = \sum_{x \in X-S} x$. PARTITION is NP-complete.

   (a) Professor P.T. Reduction believes SWD is NP-hard. To prove this he proposes the following transformation function T from a PARTITION input to a SWD input. Given $X = \{x_1, \ldots, x_n\}$, first compute $B = \sum_{i=1}^{n} x_i$. The input for SWD is as follows. There will be $n+1$ events. For event $1 \leq i \leq n$, let $s_i = 0$, $\ell_i = x_i$ and $f_i = B + 1$. Finally, he includes a $(n+1)^{st}$ event which will serve as a divider to split the other events into two sets. For this event, set $s_{n+1} = 0$, $\ell_{n+1} = 1$ and $f_{n+1} = B + 1$. Does this reduction satisfy the requirements to show that SWD is NP-hard? If not, state exactly which required condition fails and prove that it fails.

   (b) Prove there is a polynomial time algorithm for SWD or prove SWD is NP-complete.

Advanced Problems

6. (10 pts) Prove that there is a polynomial time algorithm A that solves SAT (i.e. given a boolean formula ϕ, A respond "yes" iff ϕ has a satisfying assignment) if and only if there is some polynomial time algorithm B that when given a boolean formula ϕ it computes a satisfying assignment for ϕ (or reports none exists).

   Be sure to show both directions for the "iff".

7. (15 pts) In the BOUNDED-DEGREE-SPANNING-TREE (BDST) problem you are given as input an undirected graph $G = (V, E)$ and a positive integer $K \leq |V| - 1$. The question is whether or not there is a spanning tree of G in which no vertex has degree more than K (i.e. each vertex has at most K edges of the spanning tree incident to it).

   Prove that BDST is NP-Complete. You must use one of the following NP-complete problems for your reduction: CIRCUIT-SAT, SAT, 3-CNF-SAT, CLIQUE, VERTEX-COVER, SUBSET-SUM, HAM-CYCLE, TSP.

Optional Challenge Problem

If you solve this open problem I will increase your final grade by one full letter grade. **Don't spend too much time on this problem unless you have time to spare.**

For the following scheduling problem either provide a polynomial time algorithm or prove the corresponding decision problem is NP-complete. You are given n events each of which has a duration of one unit where no two events can run at the same time. Event i is specified by two real numbers, an earliest start time s_i and a maximum delay d_i. (Thus event i can be started at time t where t is any real number satisfying $s_i \leq t \leq s_i + d_i$.) Your goal is to schedule as many events as possible. In the decision version you are also given an integer k and asked if it is possible to schedule k of the events.