**CS 241 Algorithms and Data Structures** **Spring Semester, 2004**

# Exactly Solving Recurrence Equations

*February 6, 2004* *Handout 2*

In this handout, we will exactly solve one recurrence for each of the cases of the master method as well as solving one recurrence that does not fit the form of the master method. For the recurrence $T(n) = aT(n/b) + f(n)$, for $n \geq 1$ a power of $b$ with termination at $T(1)$, the exact solution is given by the summation:

$$T(n) = \left( \sum_{i=0}^{(log_b n)-1} a^i f\left(\frac{n}{b_i}\right) \right) + T(1)a^{\log_b n}$$

This is because at level $i$ of the recursion tree (where $i = 0$ at the root and $i = \log_b n$ at the leaves), there are $a^i$ nodes. Except for the leaves $f(\frac{n}{b^i})$ time is spent excluding the recursive calls and $T(1)$ time is spent per leaf. We make use of the following summations:

- Arithemtic sum: $\sum_{i=1}^{n} i = n(n+1)/2$
- Geometric series: $\sum\limits_{i=0}^{r} x^i = (x^{r+1} - 1)/(x-1) = (1 - x^{r+1})/(1-x)$ for $x \neq 1$.
- $\sum_{i=0}^{r} i \cdot x^i = (x^{r+1}(r(x-1)-1) + x)/(x-1)^2$ which is obtained by differentiating both sides of the geometric series with respect to $x$ and then multiplying both sides by $x$. So for $x = 2$, we have $\sum_{i=0}^{r} i \cdot 2^i = 2^{r+1}(r-1) + 2$.

**Master Method: Case 1 Example.** Here we consider $T(n) = 4T(n/2) + 3n\log_2 n$, $T(1) = 1$ for $n \geq 1$ a power of 2. Applying the above general form for the summation gives

$$\begin{aligned}
T(n) &= \left( \sum_{i=0}^{(\log_2 n)-1} 4^i \left(3 \cdot \frac{n}{2^i} \log_2 \frac{n}{2^i}\right) \right) + T(1) \cdot 4^{\log_2 n} \\
&= \left( 3n \sum_{i=0}^{(\log_2 n)-1} 2^i (\log_2 n - i) \right) + n^2 \\
&= \left( 3n\log_2 n \sum_{i=0}^{(\log_2 n)-1} 2^i \right) - \left( 3n \sum_{i=0}^{(\log_2 n)-1} i \cdot 2^i \right) + n^2 \\
&= 3n\log_2 n(2^{\log_2 n} - 1) - 3n\left(2^{\log_2 n}(\log_2 n - 2) + 2\right) + n^2 \\
&= 3n^2\log_2 n - 3n\log_2 n - 3n^2\log_2 n + 6n^2 - 6n + n^2 \\
&= 7n^2 - 3n\log_2 n - 6n = \Theta(n^2)
\end{aligned}$$

You can easily check via examples or prove via induction that this formula is correct.

**Master Method: Case 2 Example.** Here we consider $T(n) = 2T(n/4) + \sqrt{n}$, $T(1) = 1$ for $n \geq 1$ a power of 4. Applying the above general form for the summation we have

$$\begin{aligned}
T(n) &= \left( \sum_{i=0}^{(\log_4 n)-1} 2^i \sqrt{n/4^i} \right) + T(1) \cdot 2^{\log_4 n} \\
&= \left( \sum_{i=0}^{(\log_4 n)-1} \sqrt{n} \right) + \sqrt{n} = \sqrt{n}\log_4 n + \sqrt{n} = \Theta(\sqrt{n})
\end{aligned}$$

You can easily check via examples or prove via induction that this formula is correct.

**Master Method: Case 3 Example.** Here we consider $T(n) = 2T(n/2) + 3n^2$, $T(1) = 2$ for $n \geq 1$ a power of 2. Applying the above general form for the summation we have

$$\begin{aligned}
T(n) &= \left( \sum_{i=0}^{(\log_2 n)-1} 2^i \cdot 3 \left(\frac{n}{2^i}\right)^2 \right) + T(1) \cdot 2^{\log_2 n} \\
&= \left( 3n^2 \sum_{i=0}^{(\log_2 n)-1} 2^i/4^i \right) + 2n \\
&= \left( 3n^2 \sum_{i=0}^{(\log_2 n)-1} (1/2)^i \right) + 2n \\
&= 3n^2 \left( \frac{1 - (1/2)^{\log_2 n}}{1 - 1/2} \right) + 2n \\
&= 6n^2 \left( 1 - \frac{1}{n} \right) + 2n \\
&= 6n^2 - 4n = \Theta(n^2)
\end{aligned}$$

You can easily check via examples or prove via induction that this formula is correct.

**A Recurrence Relation not of the Master Method Form.** Here we consider the recurrence equation $T(n) = T(n-2) + 2n$, $T(0) = 1$ for $n \geq 0$ an even integer. In this recurrence tree, at the $i$th level the problem will be of size $n - 2i$ with the root having $i = 0$ and the leaf have $i = n/2$ (so $n - 2i = 0$). For the internatl nodes the cost at level $i$ is $2i$ and for the leaf the cost is 1. Finally, there is just one node at each level of the tree. So we have

$$\begin{aligned}
T(n) &= \left( \sum_{i=0}^{(n/2)-1} 1 \cdot 2(n-2i) \right) + T(0) \\
&= \left( \sum_{i=0}^{(n/2)-1} 2n - 4i \right) + 1 \\
&= 2n \cdot (n/2) - \left( \sum_{i=0}^{(n/2)-1} 4i \right) + 1 \\
&= n^2 - \left( 4 \cdot \frac{1}{2} \left( \frac{n}{2} - 1 \right) \frac{n}{2} \right) + 1 \\
&= n^2 - 2 \left( \left( \frac{n}{2} - 1 \right) \frac{n}{2} \right) + 1 \\
&= \frac{n^2}{2} + n + 1 = \Theta(n^2)
\end{aligned}$$

Let me demonstrate how to use some examples as a "sanity check" to see that this appears to be right. We know from the recurrence that $T(0) = 1$. The formula yields 0+0+1=1 which is correct. From the recurrence we have that $T(2) = T(0) + 4 = 1 + 4 = 5$. The formula yields 4/2 + 2 + 1 = 5. From the recurrence we have $T(4) = T(2) + 8 = 5 + 8 = 13$. The formula yields $T(4) = 16/2 + 4 + 1 = 13$. Of course, the above does not prove the formula will be correct for all even $n \geq 0$ – To prove that you would use induction.