

## Common Errors in Writing Inductive Proofs

October 4, 2000

Handout 6

This handout is designed to help you improve the style of your inductive proofs. First of all, remember to always clearly indicate when you are going to apply the inductive hypothesis. Secondly, when you decompose “ $P(n + 1)$ ” to get “ $P(n)$ ” you must explain why this is a valid decomposition.

In the past, I have seen students who choose a way to decompose  $P(n + 1)$  just because it made the algebra work out. This is not a proof — you can often decompose it so the algebra would work out (even if the theorem was false). The key is that the decomposition is correct based on other reasons. To make this discussion more concrete, let’s consider the following problem.

Use mathematical induction to prove that with  $n$  different ice cream flavors to choose from there are  $n(n - 1)/2$  possibilities for a double scoop with two different flavors whenever  $n$  is an integer greater than or equal to 2.

Here is a correct proof. I will use  $f_n$  to denote the number of two scoop combinations possible using  $n$  flavors of ice cream where the two scoops must be different flavors.

**Theorem:**  $\forall n \geq 2, f_n = \frac{n(n-1)}{2}$ .

**Proof:** By mathematical induction on  $n$ .

*Basis Step:*  $f_2 = 2/2 = 1$  is correct since you must use both flavors and hence there is only one choice.

*Inductive Step:* We must show that  $\forall n \geq 2 (f_n = (n(n - 1))/2) \rightarrow (f_{n+1} = ((n + 1)n)/2)$ . So we assume if there are  $n$  ice cream flavors then there are  $n(n - 1)/2$  choices. We now consider when there are  $n + 1$  ice cream flavors. What are the choices for a double scoop? There are all the choices that existed with the first  $n$  flavors (i.e. we can apply the inductive hypothesis) plus there are the additional choices that use the  $(n + 1)$ st flavor. Notice that there are  $n$  choices that use the  $(n + 1)$ st flavor (one for each other flavor to put with it).

Thus we get that

$$f_{n+1} = f_n + n = \frac{n(n - 1)}{2} + n = n \left( \frac{n - 1}{2} + 1 \right) = n \left( \frac{n + 1}{2} \right) = (n + 1)(n)/2$$

as desired. Thus  $\forall n \geq 2 P(n) \rightarrow P(n + 1)$ .

Thus by the principle of mathematical induction we have that  $\forall n \geq 2, f_n = \frac{n(n-1)}{2}$ . ■

I have seen many incorrect proofs for this problem that base their proof on the fact that:

$$\frac{n(n-1)}{2} \cdot \frac{(n+1)}{(n-1)} = \frac{(n+1)n}{2} \quad (1)$$

If you were going to base a proof on this then you must argue that the number of two scoop choices with  $n+1$  flavors is the number of two scoop choices with  $n$  flavors (which is  $n(n-1)/2$  by the inductive hypothesis) times  $(n+1)/(n-1)$ . We know the algebra works out to give the desired result of  $((n+1)n)/2$ , but the key is WHY would the number of two scoop choices with  $n$  flavors would be given by this formula. I don't see any understandable way to argue this. Let me demonstrate that even if you have a WRONG theorem, you can always find some algebraic operation that would give the answer. For example suppose that you tried to prove that there were  $n^2$  choices (which is wrong). You could say that:

$$n^2 \cdot \frac{(n+1)^2}{n^2} = (n+1)^2$$

but I hope everyone sees that this does not prove that there are  $n^2$  choices.

Remember the key is that you *must explain why* the decomposition you give is correct and this depends on thinking about the particular problem. So let's return to the problem of showing there are  $n(n-1)/2$  two scoop choices. If in your proof you use:

$$\frac{n(n-1)}{2} + n = \frac{(n+1)n}{2}$$

then by itself it is really no better than (1) above. The key to why this is a good decomposition is that the number of two scoop choices with  $n+1$  flavors is the number of two scoop choices with  $n$  flavors (which is  $n(n-1)/2$  by the inductive hypothesis)  $+n$ . You MUST explain this. With  $n+1$  flavors, you have all the choices you did with  $n$  flavors, plus you can make a double scoop with the new flavor combined with any of the  $n$  other flavors. Thus there are  $n$  additional choices available when the  $(n+1)$ st flavor is added. Notice that the formula given above comes from observing that there are  $n$  additional two-scoop choices, NOT from just finding something so that the algebra works out.

If you do not understand the above, PLEASE come see us.

## Advice on Proof Style

Let me now talk a little about a style issue.

### A Bad Style

Let  $P(n)$  be the proposition that  $\sum_{j=0}^n (j+1) = \frac{(n+1)(n+2)}{2}$ . I'll assume the base has already been shown. Namely,  $P(0)$  is true since  $\sum_{j=0}^0 (j+1) = 1$  and  $(1 \cdot 2)/2 = 1$ . What I'll focus on is the inductive step. Namely, proving that  $\forall n \geq 0 P(n) \rightarrow P(n+1)$ .

I will now show you the style of proof of  $\forall n \geq 0 P(n) \rightarrow P(n+1)$  that I *strongly discourage you to use*.

Assume that  $P(n)$  is true (i.e.  $\sum_{j=0}^n (j+1) = \frac{(n+1)(n+2)}{2}$ ). Then

$$\begin{aligned} \sum_{j=0}^{n+1} (j+1) &= ((n+1)+1)((n+1)+2)/2 \\ \sum_{j=0}^n (j+1) + (n+1) + 1 &= (n+2)(n+3)/3 \\ \frac{(n+1)(n+2)}{2} + n + 2 &= (n+2)(n+3)/2 && \text{by the inductive hypothesis} \\ \frac{n+2}{2}(n+1+2) &= (n+2)(n+3)/2 \\ (n+2)(n+3)/2 &= (n+2)(n+3)/2 \end{aligned}$$

Since the right-hand and left-hand sides are equal it follows that  $\forall n \geq 0 P(n) \rightarrow P(n+1)$ .

### What is wrong with this style?

- **It is very confusing to the reader of the proof.** First of all, the top line shows what you would like to prove, but at first glance it appears as if you are assuming that  $P(n+1)$  is true. Furthermore, in this proof you are simultaneously working on the left and right hand sides and really between steps you are using algebra and the assumption that  $P(n)$  is true to independently rewrite the left and right hand sides. However, from the presentation this is not clear. (And even if it were, this is still not a good style.)
- **It introduces the potential for error.** Although this is an extreme example, suppose that you choose to multiply both the left and right sides by zero at some point. Then you could conclude that the left-hand side equals the right-hand side even if they were not originally equal. This would lead to an erroneous proof. When working with inequalities, the possibility of introducing an error of this form is even higher.

### The Recommended Style

When you would like to show that two expressions are equal, always start with one expression and manipulate it (legally, of course) to derive the other expression. Also don't forget to clearly indicate where you apply the inductive hypothesis. So for example, here is a well written proof that  $\forall n \geq 0 P(n) \rightarrow P(n+1)$ .

Assume that  $P(n)$  is true (i.e.  $\sum_{j=0}^n (j+1) = \frac{(n+1)(n+2)}{2}$ ). Then

$$\begin{aligned}
 \sum_{j=0}^{n+1} (j+1) &= \sum_{j=0}^n (j+1) + ((n+1)+1) && \text{splitting the summation} \\
 &= \frac{(n+1)(n+2)}{2} + (n+2) && \text{by the inductive hypothesis} \\
 &= \frac{n+2}{2}(n+1+2) && \text{factoring out an } (n+2)/2 \\
 &= \frac{(n+2)(n+3)}{2} && \text{by algebra} \\
 &= \frac{((n+1)+1)((n+1)+2)}{2} && \text{by algebra}
 \end{aligned}$$

Thus we have shown that  $\sum_{j=0}^{n+1} (j+1) = \frac{((n+1)+1)((n+1)+2)}{2}$  which is exactly what is required for  $P(n+1)$  to be true. Thus  $\forall n \geq 0 P(n) \rightarrow P(n+1)$ .

Note that when you are working on writing such a proof it is often convenient to initially work with both sides. Then when writing up the proof you need just reverse the steps you took when manipulating one of the sides. By doing this you can easily write up your proof so that you take one expression and from it derive the other expression and you will make it much less likely that you introduced an error. For example if you multiplied both sides by 0 when working with both sides, you're not going to be able to convert that to a proof that derives one expression from the other.