

## Proof by Contradiction

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Handout 3

In this handout I will demonstrate two acceptable ways (for this class) to write-up a proof by contradiction to show that an argument is valid. There are many variations of these (or those that are between the two extremes shown) that are also valid. The most important thing is that at each step of your proof that (1) there is a very clear indication as to which propositions (from the hypothesis or from conclusions drawn in prior steps of the proof) are being combined to reach the current conclusion being drawn, and (2) that you are using one of the given rules of inference or one that you have *proven* is valid.

Consider the following problem. Use a proof by contradiction to show that the following argument is valid.

$$\begin{array}{l}
 (1) \quad p \rightarrow t \\
 (2) \quad q \rightarrow s \\
 (3) \quad r \rightarrow (s \wedge t) \\
 (4) \quad p \vee q \vee r \vee u \\
 \hline
 \therefore s \vee t \vee u
 \end{array}$$

Here's proof that this is valid using the a proof by contradiction in the more formal style:

$$\begin{array}{ll}
 (5) \quad \neg(s \vee t \vee u) & \text{Negation of conclusion} \\
 (6) \quad \neg s \wedge \neg t \wedge \neg u & (5), \text{DeMorgan} \\
 (7) \quad \neg s & (6), \text{simplification} \\
 (8) \quad \neg t & (6), \text{simplification} \\
 (9) \quad \neg u & (6), \text{simplification} \\
 (10) \quad \neg q & (2), (7), \text{Modus tollens} \\
 (11) \quad \neg p & (1), (8), \text{Modus tollens} \\
 (12) \quad \neg s \vee \neg t & (7), \text{amplification} \\
 (13) \quad \neg(s \wedge t) & (12), \text{DeMorgan} \\
 (14) \quad \neg r & (3), (13), \text{Modus tollens} \\
 (15) \quad \neg p \wedge \neg q \wedge \neg r \wedge \neg u & (11), (10), (14)(9), \text{amplification} \\
 (16) \quad \neg(p \vee q \vee r \vee u) & (15), \text{DeMorgan} \\
 (17) \quad \therefore s \vee t \vee u & (4), (5), (16), \text{Contradiction}
 \end{array}$$

Here is the proof by contradiction in a more English style argument:

For the conclusion to be false we need  $s = t = u = F$ . From  $p \rightarrow t$  and  $t = F$ , by modus tollens  $p = F$ . From  $q \rightarrow s$  and  $s = F$  by modus tollens  $q = F$ . Since  $s = F$  and  $t = F$ ,  $(s \wedge t) = F$ . And since  $(s \wedge t) = F$  and  $r \rightarrow (s \wedge t)$ , by modus tollens  $r = F$ . But then since  $p = q = r = u = F$ ,  $p \vee q \vee r \vee u$  is  $F$  contradicting (4). Thus the given argument is valid.

Once again, it is really important that at each step there is an explicit indication as to which propositions (from the hypothesis of the argument or from conclusions drawn in prior steps of the proof) are being combined to reach the conclusion being drawn. Also, be sure that you are using one of the given rules of inference or one that you have proven is valid. In the more formal style, the statement numbers are given to indicate which propositions are being combined (and the rule of inference is explicitly named). In the second style given, the prior steps being combined are re-stated. While the rule of inference is not explicitly given, it should be easy for a reader familiar with these rules to see which is being applied. If you are using a new rule of inference, you must prove that it is valid. Either of these styles (or one combining aspects of each) is fine *for this course*.