

## Homework Assignment 9

November 15, 20

Due Date: November 29 (Quiz on Nov. 27)

**NOTE: On the exercises on counting, you must clearly explain your reasoning to receive full credit. Just the answer without any explanation is not enough.**

**Practice Exercises**

1. How many different three letter initials are there when
  - (a) all of the initials are different
  - (b) the first and last initials are the same (the middle initial is not restricted)
  - (c) at least 2 different letters are used
2. How many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, including the bride and groom, if
  - (a) the bride must be in the picture
  - (b) both the bride and the groom must be in the picture
  - (c) exactly one of the bride and the groom is in the picture
3. Give a combinatorial proof for the following:  $C(2n, 2) = 2C(n, 2) + n^2$  where  $n$  is a nonnegative integer.  
*Hint: Think about two ways to count the numbers of possibilities for selecting two people from  $n$  men and  $n$  women.*
4. Consider an error-correcting code (each string in the code is just a bit string) in which any legal string has no more than 3 consecutive 1s. Prove that all sets of 200 legal strings in this code contain at least 7 beginning with the same 5 bits as each other.
5. Suppose there are 6 types of apples at the store. You want to buy 8 apples with at least 2 golden delicious apples and at most 3 red delicious apples. How many ways are there to do this when what matters is how many apples you buy of each type?
6. How many different strings can be made from the letters in MISSOURI when
  - (a) using all the letters?
  - (b) using all the letters when the two “S”s must be consecutive?

**Problems to Submit**

1. (10 pts) What is the probability of being dealt a “two pair” hand in poker (from a standard deck of 52 cards that has been shuffled)? A “two pair” hand consists of two cards of one rank, two cards of another rank, and a final card of a third rank. So for example a hand with two kings, two 5s and an 8 is a “two pair” hand.
2. (10 pts) Consider a computer network consisting of  $n$  computers (for  $n \geq 2$ ) where each computer is directly connected to zero or more of the other computers. Prove that there are at least two computers in the network that are directly connected to the same number of other computers.  
*Hint: You may find it useful to first think about the special case in which there are, say 6, computers on the network. Can you have one computer connected to no others and another connected to 5 others? Now generalize your ideas to get a proof for the the general case of  $n$  computers.*

3. (15 pts) Give combinatorial proofs for each of the following:
- (a)  $C(n, r) \cdot C(r, k) = C(n, k) \cdot C(n - k, r - k)$  whenever  $n$ ,  $r$ , and  $k$  are nonnegative integers with  $r \leq n$  and  $k \leq r$ .  
*Hint: Think about two ways to count the number of possibilities for selecting  $r$  people from  $n$  who tried out for a team as well as selecting  $k$  of those  $r$  as the starters.*
- (b)  $\sum_{i=1}^n (i \cdot C(n, i)) = n2^{n-1}$ .  
*Hint: Think about number of possibilities for selecting a committee from  $n$  people (of size between 1 and  $n$ ) for which there is a designated leader of that committee.*
4. (15 pts) A bagel shop has onion, poppy seed, egg, salty, pumpernickel, sesame seed, raisin, and plain bagels. How many ways are there to choose
- (a) a dozen bagels?  
 (b) a dozen bagels with at least one of each kind?  
 (c) a dozen bagels with between 3 and 6 egg bagels and no more than three salty bagels?

**Challenge Problem: Let's Make a Deal**

You are a contestant in a game show and in front of you are three curtains and behind one of them is a prize that is equally likely to be behind each of the curtains. Suppose at this point you get to pick a curtain and will win the prize if you have selected the correct one. But the game does not end here. Next the game show host picks one of the two curtains that you did not pick and opens it to show you that there is no prize there. Before the curtain you picked is opened you are offered the opportunity to switch to the other curtain which has not yet been opened.

If you want to maximize your chance of winning, should you switch? Why or why not? What is your probability of winning? Carefully explain your reasoning. This is a problem in which many people's intuition is often wrong!

We now add a new twist in the game. If you are wrong, With probability  $p$  the game show host just opens the curtain you picked without giving you the option to switch. What is the probability of winning if you switch (as a function of  $p$ )? Then what value of  $p$  would be selecting if the probability of winning is equal whether or not you switch.