

## Homework Assignment 8

November 8, 2000

Due Date: November 15 (Quiz on Nov. 13)

**Practice Exercises**

1. Let  $f(x) = 2x + 1$  and let  $g(x) = (x - 3)^2$ . For each function given below you should answer the following questions. Is the function an injection? Is the function a surjection? Is the function a bijection? If the function is a bijection then give its inverse. You should briefly explain your answers.
  - (a) The function  $f$  when the domain and the codomain are the reals.
  - (b) The function  $f$  when the domain is the integers and the codomain is the reals.
  - (c) The function  $f \circ g$  when the domain and codomain are the reals. (You should first compute  $(f \circ g)(x) = f(g(x))$ .)
2. Let  $g$  be a function from  $X$  to  $Y$ , and let  $f$  by a function from  $Y$  to  $Z$ . Prove or disprove each of the following:
  - (a) If  $f$  and  $f \circ g$  are surjective, then  $g$  is surjective.
  - (b) If  $g$  is not injective, then  $f \circ g$  is not injective.
3. Give a closed form solution for  $\sum_{k=2}^{n-3} (3k + 2^k)$ .
4. In this problem we go over sets.
  - (a) Let  $A = \{1, \{1\}, \{2\}\}$ . Which of the following statements are true?  
 $\{1\} \in A$ ,  $1 \subseteq A$ ,  $\{1, 1\} \subseteq A$ ,  $\{2\} \subseteq A$ ,  $\{\{1\}\} \in A$ .
  - (b) For  $S = \{1, a, b\}$  give the powerset  $P(S)$ .
  - (c) Prove or disprove: If  $A \subseteq B$  then  $P(A) \subseteq P(B)$ .
5. Let  $A = \{1, 2, \dots, n\}$  where  $n$  is a positive integer, and let  $B = \{a, b\}$ .
  - (a) How many functions are there from  $A$  to  $B$ ?
  - (b) How many functions are there from  $A$  to  $B$  for which exactly one element from the domain maps to "a".
  - (c) How many surjective functions are there from  $A$  to  $B$ ?
6. Prove whether or not the set  $S$  of all bit strings is countable. (A bit string is defined as a string of bits that has some finite length.) Be sure to clearly indicate if you are proving it is countable or uncountable.

**Problems to Submit**

1. (15 pts) Let  $g$  be a function from  $X$  to  $Y$ , and let  $f$  by a function from  $Y$  to  $Z$ . Prove or disprove each of the following statements.
  - (a) If  $f$  is not injective then  $f \circ g$  is not injective.
  - (b) If  $f$  is not surjective then  $f \circ g$  is not surjective.
  - (c) If  $f$  and  $f \circ g$  are both injective, then  $g$  is injective.

2. (10 pts) Prove or disprove each of the following. You may use any prove technique *except* a membership table or Venn Diagram.
- (a) Prove or disprove:  $A - (B \cap C) = (A \cap \overline{B}) \cup (A \cap \overline{C})$ .
  - (b) Prove or disprove:  $P(A - B) = P(A) - P(B)$ .
3. (15 pts) Here we focus on the limitations of what a computer can compute.
- (a) Prove that the set of all computer programs in *any* given programming language is countable.  
*Big hint: A computer program in a given programming language can be thought of as a string of symbols from a finite alphabet. Let  $k$  be the number of symbols in the alphabet for the given language and then you can think of any program in this language as a base- $k$  number.*
  - (b) Consider the set  $F$  of functions from the natural numbers to  $\{0, 1\}$ . (You can think of the domain as the set of possible inputs to a program that answers a yes/no question.) Prove that  $F$  is uncountable.
  - (c) Using your answer from parts (a) and (b) to respond the the following question (explaining your reasoning).  
 Are there any problems for which there does *not* exist any computer program (with no restriction that it be efficient) to solve the problem?
4. (10 pts) In a standard combination lock, the combination is defined by a 1st, 2nd and 3rd number where each of these three numbers is a unique integer between 0 and 39 (i.e. a number cannot be used twice in the combination). Consider a “deluxe combination lock” in which the combination can consist of 2, 3 or 4 numbers (with the rest being as described above).  
 How many possibilities are there for this deluxe combination lock?

### Challenge Problem

Here we explore Russell’s paradox.

- (a) There is a barber who will shave those and only those who do not shave themselves. Who will shave the barber?
- (b) Now let’s relate part (a) to set theory. Let  $S$  be a set of sets such that  $S = \{x | x \notin x\}$ . Is  $S \in S$  or is  $S \notin S$ ? Why?
- (c) What occurred in part (b) is known as Russell’s paradox. How can you restrict the method of defining sets so that this paradox is avoided (yet, the type of sets that we would like to define can still be defined)?