

## Homework Assignment 7

October 25, 2000

Due Date: November 1 (Quiz on Oct. 30)

Remember these are the steps you must follow when proving an iterative program is correct:

- Clearly state the loop invariant “ $p$ ” that you will use.
- Use induction to prove that  $p$  is a loop invariant. (So for the base step you prove that  $p$  is true just before the loop is first entered, and for the inductive step you prove  $(p \wedge \text{cond})\{M_L\}p$  where  $M_L$  is the loop body and  $\text{cond}$  is the condition.)
- Now use your loop invariant to prove that the given program is partially correct with respect to the given initial and final assertions.
- Finally, prove the given program is correct with respect to the given initial and final assertions (i.e. show that it always terminate).

Practice Exercises

1. Use a loop invariant to prove the correctness of the following algorithm to compute the maximum of a sequence of integers  $a_0, \dots, a_{n-1}$  with respect to the initial assertion  $n \geq 1$  and the final assertion that  $\text{ans} = \max\{a_0, \dots, a_{n-1}\}$ .

```

procedure computeMax( $\{a_0, \dots, a_{n-1}\}$ )
   $\text{ans} = a_0$ 
   $i = 1$ 
  while ( $i < n$ )
    if  $a_i > \text{ans}$  then  $\text{ans} = a_i$ 
     $i = i + 1$ 
  return  $\text{ans}$ 

```

2. For the following program: (1) give a meaningful loop invariant  $p$ , and (2) assuming that  $p$  is a loop invariant (you need not prove that for this problem), argue that the program is correct under the initial assertion that that  $n$  is a positive integer.

```

procedure factorial( $n$ )
   $\text{ans} = 1$ 
   $i = 2$ 
  while ( $i \leq n$ )
     $\text{ans} = \text{ans} * i$ 
     $i = i + 1$ 
  return  $\text{ans}$ 

```

3. Prove that the loop invariant that you gave in practice problem 2 is a loop invariant. (For the quiz, I will write up the loop invariant for you if this problem is selected.)

4. For the following program: (1) give a meaningful loop invariant  $p$ , and (2) assuming that  $p$  is a loop invariant (you need not prove that for this problem) prove that the program is correct under the initial assertion that  $n$  is a positive integer.

```

procedure mult( $n, m$ )
   $prod = 0$ 
   $k = n$ 
  while ( $k > 0$ )
     $k = k - 1$ 
     $prod = prod + m$ 
  return  $prod$ 

```

5. Prove that the loop invariant that you gave in practice problem 4 is a loop invariant. (For the quiz, I will write up the loop invariant for you if this problem is selected.)
6. Consider the following buggy program to compute  $ans = n(n + 1)/2$  with respect to the initial assertion that “ $n$  is a positive integer” (i.e.  $n \geq 1$ ). It was designed using loop invariant  $p$ : “ $(ans = i(i + 1)/2) \wedge (i \leq n)$ .” However, as written,  $p$  is not a loop invariant. Modify the program so that  $p$  is truly a loop invariant and then prove it is correct.

```

procedure sum( $n$ )
   $ans = 0$ 
   $i = 0$ 
  while ( $i \leq n - 1$ )
     $ans = ans + i$ 
     $i = i + 1$ 
  return  $ans$ 

```

### Problems to Submit

1. (15 pts) Prove the correctness of the following iterative program for finding the  $n$ th Fibonacci number with respect to the initial assertion  $n \geq 0$  and the final assertion that  $y$  is the  $n$ th Fibonacci number,  $f_n$ . Recall  $f_0 = 0$ ,  $f_1 = 1$  and  $\forall n \geq 2, f_n = f_{n-1} + f_{n-2}$ .

```

procedure fibonacci( $n$ )
  if ( $n == 0$ ) then  $y = 0$ 
  else
     $x = 0$ 
     $y = 1$ 
    for  $i = 1$  to  $n - 1$ 
       $z = x + y$ 
       $x = y$ 
       $y = z$ 
  return  $y$ 

```

2. (15 pts) Use a loop invariant to prove that the following program is correct with respect to the initial assertion “ $a$  and  $d$  are positive integers” and the final assertion “ $q$  and  $r$  are integers such that  $a = dq + r$  and  $0 \leq r < d$ .”

```
procedure divide( $a, d$ )  
   $r = a$   
   $q = 0$   
  while ( $r \geq d$ )  
     $r = r - d$   
     $q = q + 1$   
  return  $\{q, r\}$ 
```

*Hint: Think about why this works. It may help to trace the procedure on a few examples. What's the relationship between  $a, d, q$ , and  $r$ ?*

3. (15 pts) Use a loop invariant to prove that the following program is correct with respect to the initial assertion that  $x$  is a positive integer and the final assertion that  $ans = x^2$ .

```
procedure square( $x$ )  
   $i = j = 1$   
  while ( $i < x$ )  
     $j = j + 2i + 1$   
     $i = i + 1$   
  return  $j$ 
```

### Challenge Problem

Use a loop invariant to prove that the following program is correct with respect to the initial assertion that  $n$  is an integer  $\geq 0$  and the final assertion that `palindrome( $n$ )` returns the integer obtained by reversing the digits in  $n$ . (For example, if  $n = 1432$  then it should return 2341.)

```
procedure palindrome( $n$ )  
   $reverse = 0$   
   $m = n$   
  while  $m > 0$   
     $temp = m \% 10$   
     $reverse = reverse * 10 + temp$   
     $m = (m - temp) / 10$   
  return  $reverse$ 
```