CS 201 Formal Foundations of Computer Science	Fall Semester, 2000

Homework Assignment 5

October 11, 2000 — *Due Date: October 18 (Quiz on Oct. 16)*

Practice Exercises

1. Use mathematical induction to show that $\neg(p_1 \vee p_2 \vee p_n)$ is logically equivalent to $\neg p_1 \wedge \neg p_2 \cdots \wedge \neg p_n$ for propositions $p_1, \ldots, p_n$ for $n \geq 2$. Note that you may only use DeMorgan's rule with two propositions (i.e. $\neg(p \vee q) \iff \neg p \wedge \neg q$).

2. What is wrong with the following proof that claims to show that for $a \neq 0$, $a^n = 1$ for all non-negative integers n?

   Basis: $a^0 = 1$ by definition of a^0.

   Inductive Step: Assume that $a^k = 1$ for all non-negative integers k such that $k \leq n$. To show that the hypothesis is true for $n+1$, we see that

   $$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1$$

   where we get the last step by applying the inductive hypothesis that $a^n = 1$ and $a^{n-1} = 1$.

3. Prove (using mathematical induction) that for any integer $n \geq 1$, n straight lines in the plane, all passing through a single point, divide the plane into $2n$ regions.

4. Use mathematical induction to prove that for n an integer one or greater, there are $n!$ ways to arrangements of the numbers $1, 2, \ldots, n$ in an array of size n.

5. After n months a certain greenhouse experiment, the number, $P(n)$ of plants satisfies

   $$\begin{aligned} P(0) &= 3 \\ P(1) &= 7 \\ P(n) &= 3P(n-1) - 2P(n-2) \text{ for all } n \geq 2 \end{aligned}$$

   Use mathematical induction to prove that $P(n) = 2^{n+2} - 1$ for all integers $n \geq 0$.

6. For a and b positive real numbers, let $P(n)$ denote that $(a+b)^n \geq a^n + b^n$. Prove by mathematical induction that for $P(n)$ is true for n any positive integer.

Problems to Submit

1. (12 pts) Use mathematical induction to show that 3 divides $n^3 + 2n$ whenever n is a non-negative integer.

2. (15 pts) Prove that n lines separate the plane into $(n^2 + n + 2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

   Hint: When you add the $(n+1)$st line, how many regions are split by the line?

3. (12 pts) Let $P(n)$ be the predicate that there are n^2 two scoop options with n flavors where both scoops can be the same flavor. Note that flavor 1 on top of flavor 2 is to be viewed as a different option than flavor 2 on top of flavor 1. Prove (using mathematical induction) that $\forall n \geq 1, \quad P(n)$.

4. (11 pts) Let a_n be defined as follows: $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, and for all $n \geq 4$, $a_n = 2a_{n-3} + a_{n-1}$.

   Let $P(n)$ be $a_n \leq 2^{n-1}$. Use mathematical induction to prove that $\forall n \geq 1\ P(n)$.

Challenge Problems

1. Use induction to prove that the sum of the (base-10) digits in a number x are divisible by 9 if and only if x is divisible by 9.
2. The city of Logicus is inhabited by citizens with the following characteristics:

   1. The citizens are at a party where they can see everyone's face except their own;
   2. No citizen ever tells another anything that would embarrass that citizen.
   3. All citizens have perfect (and instantaneous) logical reasoning abilities.

   Brian crashes the party and announces that at least one person has poppy seeds in their teeth. After his proclamation, Brian immediately leaves. The citizens (not having a mirror) decide that every 5 minutes (where all citizens look at the same clock) whoever discovers they have poppy seeds in their teeth will excuse themself from the party (and we'll assume that nobody would ever dream of leaving the party for any other reason).

   Recall that, in summary we have that, nobody directly knows if they have poppy seeds in their teeth, because nobody would tell them this; however, everyone can see who else has poppy seeds in their teeth.

   Let n denote the number of people at the party who have poppy seeds in their teeth. (Remember that the citizens only know that $n \geq 1$ and are not directly given any other information about its value.)

   (a) When, if ever, will those with poppy seeds on their teeth leave the party?

   (b) Use a form of mathematical induction to prove that your answer is correct.