CS 201 Formal Foundations of Computer Science	**Fall Semester, 2000**

Homework Assignment 4

October 4, 2000 — *Due Date: October 11 (Quiz on Oct. 9)*

Note: For anyone who will not be attending class on October 9th because of Yom Kippur, please send me an email to arrange an alternate date for the quiz.

Practice Exercises

1. For the following argument, prove whether or not it is valid. *Hint: Think about trying a proof by contradiction.*

   $$\begin{array}{ll} (1) & \forall x\ (\neg R(x) \rightarrow \exists y\ \neg P(x,y)) \\ (2) & \forall x\ (Q(x) \rightarrow \neg\forall y\ P(x,y)) \\ (3) & (\forall x\ \neg S(x))\ \rightarrow\ (\exists z\ Q(z)) \\ \hline & \therefore\ \exists y\ (R(y) \vee S(y)) \end{array}$$

2. Prove or disprove that the following is valid for n an integer.

   $$(n+1)^2 \text{ is even if and only if } n \text{ is odd.}$$

   Remember you must prove the following two things: if $(n+1)^2$ is even then n is odd, and if n is odd then $(n+1)^2$ is even.

3. Prove or disprove each of the following:
   (a) The sum of any six consecutive integers is divisible by 6.
   (b) The sum of any seven consecutive integers is divisible by 7.
   (c) If i divides n^3 (in other words n^3/i is an integer), then i divides n.
   (d) If i divides n, then i divides n^3.

4. Suppose your goal is to prove that propositions a, b, c, d, e, and f are logically equivalent. Suppose you have already shown the following implications are true:
   $b \rightarrow d$, $c \rightarrow a$, $d \rightarrow e$, $d \rightarrow b$, $a \rightarrow f$.
   You would like to prove the fewest additional implications as you can. List the additional implications that you would prove.

5. Let $P(n)$ be the proposition that $\sum_{i=1}^{n} i^3 = [n(n+1)/2]^2$. Use mathematical induction to prove that $\forall n \geq 1,\ P(n)$.

6. Let $P(n)$ be that any n lines, where no two are parallel, divide the plane into n^2+1 regions. What is wrong with the following inductive proof? It is not sufficient to give a counterexample to the given thereom. Rather, you must find the flaw in the proof.

   Theorem: $\forall n \geq 1,\ P(n)$

   Proof: By mathematical induction on n.
   Basis Step: 1 line divides the plane into 2 regions and $1^2+1=2$. Hence $P(1)$ is true.
   Inductive Step: We must show that $\forall n \geq 1\ P(n) \rightarrow P(n+1)$. By the inductive hypothesis there are n^2+1 regions formed with n lines. Note that $(n+1)^2+1 = n^2+1+2n+1$. So adding the $(n+1)$st line creates $2n+1$ new regions. Hence the number of regions with $n+1$ lines is $n^2+1+2n+1 = (n+1)^2+1$
   Since $P(1)$ is true and $\forall n \geq 1,\ (P(n) \rightarrow P(n+1))$, by the principle of mathematical induction we have that $\forall n \geq 1,\ P(n)$. ■(end of proof)

Problems to Submit

1. (8 pts) Let $x_1, x_2, \ldots, x_n$ be n real numbers. Let $\overline{x} = (x_1 + x_2 + \cdots + x_n)/n$ be their average. Use a proof by contradiction to prove that at least one of $x_1, x_2, \ldots, x_n$ is less than or equal to $\overline{x}$.
2. (15 pts) Prove that for n an integer then the following three statements are equivalent:
   (a) $n/5$ has a remainder of 0.
   (b) $n^2/5$ has a remainder of 0.
   (c) $n^2/5$ does not have a remainder of 1 or 4.
3. (12 pts) Prove that for all configurations of four points on a piece of paper, there exists a way to color the points (where each point must be colored red or blue) such that there is *no* line for which all the blue points are on one side of the line and all the red points are on the other side of the line.

   Note: For those of you interested, I can explain how this problem when generalized to higher dimensions, is part of a very interesting result about the amount of data needed to train a simple form of a neural network. The red and blue points represent different kinds of data that you want to distinquish between.

   Let me help you get started. Use a proof by cases.

   Case 1: There are at least 3 collinear points. Let's call those points p_1, p_2, p_3 where p_2 is between p_1 and p_3. Notice if p_1 and p_3 are colored red, and point p_2 is colored blue (with the fourth point colored arbitrarily) then no line separates the red points from the blue points. Any line that separates p_1 from p_2 will place p_3 on the same half with p_2 yet it belongs on the half with p_1.

   You take it from here ....
4. (15 pts) <u>Use mathematical induction</u> to prove that if you are having a party of $k \geq 8$ couples (so $2k$ people) and you have an unlimited supply of tables that seat 6 people and an unlimited supple of tables that seat 8 people then you can select tables so that all tables are full (i.e. you will have $2k$ seats).

Challenge Problems

1. Suppose you have a set of homes that you want to connect via a communications network. Assume that you can directly place a connection between any two homes with a cost that is proportional to the distance between the two homes. Your goal is to choose which set of connections to install so that you incur the minimum possible cost under the constraint that you must ensure that all homes are connected (i.e. between any two homes you can follow a sequence of connections from one to the other)

   Prove that there exist a solution to this problem that is optimal (i.e. it connects all the homes and has the minimum possible cost) which puts a direct connection between the two closest homes.
2. Use Problem 1 to help show that if the first 10 positive integers are placed around a circle in any order, there exist three integers in consecutive locations around the cirlce that have a sum greater than or equal to 17.