

## Homework Assignment 10

November 29, 2000

Due Date: December 6 (Quiz on Dec 4)

Practice Exercises

- For each relation given below, indicate whether the binary relation  $R$  over  $A$  (i.e.  $R$  is a relation from  $A$  to  $A$ ) is reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and transitive. You should briefly explain why it has or does not have each property.
  - $A$  is the set of lines in the plane,  $R = \{(a, b) \mid a \text{ and } b \text{ are parallel (or coincident)}\}$
  - $A = \{0, 1, 2, \dots\}$  is the set of natural numbers,  $R = \{(x, y) \mid x = y^2\}$
  - $A$  is the set of all people,  $R = \{(a, b) \mid a \text{ and } b \text{ have the same (biological) mother and father}\}$
- Count (and explain) the number of relations there are over a set  $A$  of  $n$  elements that are:
  - symmetric and reflexive
  - antisymmetric
  - asymmetric
- Let  $R$  and  $S$  be relations over a set  $A$ . Prove or disprove each of the following statements.
  - If  $R$  and  $S$  are irreflexive, then  $R \cup S$  is irreflexive.
  - If  $R$  and  $S$  are asymmetric, then  $R \cup S$  is asymmetric.
  - If  $R$  and  $S$  are asymmetric, then  $R \cap S$  is asymmetric.
- Let  $R$  be the relation  $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$  and let  $S$  be the relation  $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$ . Find  $S \circ R$ .
- Prove or Disprove. Let  $R$  be a relation on the set  $A$ . Prove that if  $R$  is reflexive then for all integers  $n$ ,  $R^n$  is reflexive.  
*Hint: If you think this is true, induction may be a good proof technique to try.*
- Let  $R = \{(a, a), (b, b), (b, c), (b, d), (b, e), (c, c), (c, d), (d, d), (e, d), (e, e), (f, f)\}$  be a relation over  $A = \{a, b, c, d, e, f\}$ . Notice that  $R$  is a partial order.
  - Show  $G_R$  the graphical view of  $R$  in which there is a vertex for each element in  $A$  and a directed edge for each element of  $R$ .
  - Show the Hasse diagram for  $R$ .
  - What are the maximal elements?
  - What are the minimal elements?

Problems to Submit

- (15 pts) Let  $R$  be a symmetric relation on a set  $A$ . Prove that  $R^n$  is symmetric for all positive integers  $n$ .

2. (15 pts) Suppose that  $R_1$  and  $R_2$  are equivalence relations on the set  $S$ . Prove or disprove each of the below statements.
- Then it follows that  $R_1 \cap R_2$  is an equivalence relation.
  - Then it follows that  $R_1 \cup R_2$  is an equivalence relation.
3. (10 pts) Let  $R = \{(a, a), (a, f), (b, c), (c, c), (d, d), (e, b), (e, e), (f, a)\}$  be a relation over the set  $A = \{a, b, c, d, e, f\}$ .
- Show  $G_R$  the graphical view of  $R$  in which there is a vertex for each element in  $A$  and a directed edge for each element of  $R$ .
  - Compute the transitive closure  $R^*$  for  $R$ . Show your work or in some way explain to us how you obtained your answer. You may show your answer in any form (graph, matrix, or as a set of tuples).
  - Is  $R^*$  is an equivalence relation? Clearly justify your answer. If you answered “yes” that  $R^*$  is an equivalence relation, clearly state the equivalence classes?
4. (10 pts) Here we study partial orders.
- Draw the Hasse diagram for the relation  $R = \{(a, b) \mid a \text{ divides } b\}$  on the set  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ . (Recall that  $a$  divides  $b$  if  $b/a$  is an integer.)
  - What are the maximal element(s)?
  - Is there a least element (explain)?
  - Which of the following pairs are comparable: 2 and 8, 3 and 8, 1 and 9, 3 and 4?

### Challenge Problem

Somebody has hidden a treasure in one of two single-story buildings (say,  $A$  or  $B$ ). To claim the treasure, you must provide a proof of which building must house the treasure. Here are your clues:

- The treasure is hidden in a room with an odd number of doors<sup>1</sup>.
- Building  $A$  has two “outside” doors, building  $B$  has three “outside” doors.
- You are now sufficiently informed to solve this puzzle.

*Hint: Can you prove that one of the two buildings must have a room with an odd number of doors?*

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<sup>1</sup>We define a door as a passageway between two rooms or between the building and the outside.