

## Supplementary Material on Logical Arguments

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Handout 4

The textbook's coverage of logical arguments with quantifiers is quite limited with only one very simple example. This handout contains some additional examples that were covered in class. Please use this handout to supplement your own notes.

**Example 1**

Consider the following: "All lions are fierce. Some lions do not drink coffee. Hence some fierce creatures do not drink coffee." Let's define the following predicates.  $L(x)$  denotes that  $x$  is a lion,  $F(x)$  denotes that  $x$  is fierce, and  $C(x)$  denotes that  $x$  drinks coffee. For all predicates the universe of discourse is all creatures.

The given statement can be expressed as the following logical argument:

$$\begin{array}{l} (1) \quad \forall x L(x) \rightarrow F(x) \\ (2) \quad \exists x L(x) \wedge \neg C(x) \\ \hline \therefore \exists x F(x) \wedge \neg C(x) \end{array}$$

We now prove that this is a valid argument:

- |  |                                 |
|--|---------------------------------|
| (3) For some $a$ , $L(a) \wedge \neg C(a)$ | (2), existential instantiation  |
| (4) $L(a)$                                 | (3), simplification             |
| (5) $\neg C(a)$                            | (3), simplification             |
| (6) $L(a) \rightarrow F(a)$                | (4), universal instantiation    |
| (7) $F(a)$                                 | (4),(6), modus ponens           |
| (8) $F(a) \wedge \neg C(a)$                | (5),(7), conjunction            |
| (9) $\exists x F(x) \wedge \neg C(x)$      | (8), existential generalization |

**Example 2**

Is the following argument valid?

$$\begin{array}{l} (1) \quad \exists x L(x) \rightarrow F(x) \\ (2) \quad \exists x L(x) \wedge \neg C(x) \\ \hline \therefore \exists x F(x) \wedge \neg C(x) \end{array}$$

It is not. Intuitively, the reason is that statement (1) and (2) may not hold for the same object. So you can create a counterexample by letting (1) hold for one object (say " $a$ "), letting (2) hold for a different object (say " $b$ "), and having the conclusion be false. However, just saying this does not prove it is false. Maybe the intuition is wrong and you can't really create such a counterexample.

Here's is a counterexample to prove that it is not valid. Let the universe of discourse consist of just  $a$  and  $b$ . Let  $L(a) = T, L(b) = T, F(a) = T, F(b) = F, C(a) = T, C(b) = F$ . Statement (1) holds when  $x = a$ , statement (2) hold when  $x = b$ , yet the conclusion does not hold for either.

### Example 3

Here's a much more involved example. Consider the following: "All hummingbirds are richly colored. No large birds live on honey. Birds that do not live on honey are dull in color. Hence hummingbirds are small." We make the following additional assumptions:

- All hummingbirds are birds.
- The negation of richly colored is dull in color.
- The negation of small is large.

Let  $B(x)$  denote that  $x$  is a bird,  $H(x)$  denote that  $x$  is a hummingbird,  $R(x)$  denote that  $x$  is richly colored,  $L(x)$  denote that  $x$  lives on honey, and  $S(x)$  denote that  $x$  is small. Then the above statement (along with the additional assumptions) yields the following argument:

$$\begin{array}{l}
 (1) \quad \forall x H(x) \rightarrow B(x) \\
 (2) \quad \forall x H(x) \rightarrow R(x) \\
 (3) \quad \neg \exists x B(x) \wedge \neg S(x) \wedge L(x) \\
 (4) \quad (B(x) \wedge \neg L(x)) \rightarrow \neg R(x) \\
 \hline
 \therefore \forall x H(x) \rightarrow S(x)
 \end{array}$$

We now show that this is a valid argument. (Note that if any of the three assumptions are removed then it is no longer valid. That is, a counterexample could be created if we were allowed to violate any of the three assumptions.)

To help give a better feel of how you create a proof that an argument is valid, we did this together in class. As discussed in class, when writing up the proof, any unnecessary statements should be removed and any new rules of inference used must be shown valid. Here is a write-up of the proof we created together in class. I will also provide a second proof just to demonstrate that there is no one right (or even best) proof.

First, we argued that when we have two universally quantified statements, that all of the rules of inference can be directly applied to them. Just to be sure everyone understands this, let me proof that the following rule (which you may just call Hypothetical Syllogism) is valid:

$$\begin{array}{l}
 (1) \quad \forall x P(x) \rightarrow Q(x) \\
 (2) \quad \forall x Q(x) \rightarrow R(x) \\
 \hline
 \therefore \forall x P(x) \rightarrow R(x)
 \end{array}$$

Here's the proof it is valid. Remember that  $U$  is the universe of discourse.

- (3)  $P(a) \rightarrow Q(a)$  for any  $a \in U$  (1), universal instantiation
- (4)  $Q(a) \rightarrow R(a)$  for any  $a \in U$  (2), universal instantiation
- (5)  $P(a) \rightarrow R(a)$  for any  $a \in U$  (3),(4), hypothetical syllogism
- (6)  $\forall x P(x) \rightarrow R(x)$  (5), universal generalization

This is *not* a rule that I feel you would be expected to prove is valid since it is really just an application of hypothetical syllogism. However, for completeness I wanted to go through such a proof once. Again, all of the rules of inference will apply if ALL statements in the hypothesis (i.e. above "the line") are *universally* quantified. You **cannot** do this if any of the statements in the hypothesis are existentially quantified. If you have any uncertainties about this please come talk to us about it.

There are two additional rules that we used. They are both very useful, so I will give them names so that you can use them later and easily refer back to them. The first, which I will call *hypothetical conjunction*, is

$$\frac{p \rightarrow q \quad p \rightarrow r}{\therefore p \rightarrow (q \wedge r)}$$

There are many ways to prove that this is valid. This easiest is to note that

$$(p \rightarrow q) \wedge (p \rightarrow r) \iff (\neg p \vee q) \wedge (\neg p \vee r) \iff \neg p \vee (q \wedge r) \iff p \rightarrow (q \wedge r).$$

That is  $[(p \rightarrow q) \wedge (p \rightarrow r)] \iff [p \rightarrow (q \wedge r)]$  and hence  $[(p \rightarrow q) \wedge (p \rightarrow r)] \rightarrow [p \rightarrow (q \wedge r)]$  making the rule of inference valid.

Another useful rule of inference, which I will call *hypothetical simplification*, is

$$\frac{p \rightarrow (q \wedge r)}{\therefore p \rightarrow q}$$

To show this is valid will use a proof by contradiction. So assume that the argument is not valid. That is, we can assert  $p \rightarrow (q \wedge r)$  and  $\neg(p \rightarrow q) \iff \neg(\neg p \vee q) \iff p \wedge \neg q$ . So by simplification, we have  $p$  and also  $\neg q$ . Combining  $p$  and  $p \rightarrow (q \wedge r)$  by modus ponens yields  $q \wedge r$ , and by simplification we have  $q$ . However, we have already said that for the conclusion to be false we must have  $\neg q$ . Hence we have reached a contradiction which shows that the argument is valid (since no counterexample can exist).

We are now ready to return to the original problem. So that everything is together, I'll start on the next page.

Here's the logical argument.

$$\begin{array}{l}
 (1) \quad \forall x H(x) \rightarrow B(x) \\
 (2) \quad \forall x H(x) \rightarrow R(x) \\
 (3) \quad \neg \exists x B(x) \wedge \neg S(x) \wedge L(x) \\
 (4) \quad (B(x) \wedge \neg L(x)) \rightarrow \neg R(x) \\
 \hline
 \therefore \forall x H(x) \rightarrow S(x)
 \end{array}$$

Here is the proof we created in class to show it was valid:

$$\begin{array}{ll}
 (5) \quad \forall x \neg(B(x) \wedge \neg S(x) \wedge L(x)) \iff \forall x \neg B(x) \vee S(x) \vee \neg L(x) & \text{logically equiv to (3)} \\
 (6) \quad \forall x B(x) \rightarrow (S(x) \vee \neg L(x)) & (5), \text{implication} \\
 (7) \quad \forall x H(x) \rightarrow (S(x) \vee \neg L(x)) & (1),(6), \text{hyp. syllogism} \\
 (8) \quad \forall x R(x) \rightarrow \neg(B(x) \wedge \neg L(x)) & \text{contrapositive of (4)} \\
 (9) \quad \forall x H(x) \rightarrow \neg(B(x) \wedge \neg L(x)) & (2),(8), \text{hyp. syllogism} \\
 (10) \quad \forall x H(x) \rightarrow (\neg B(x) \vee L(x)) & (9), \text{DeMorgan} \\
 (11) \quad \forall x H(x) \rightarrow (B(x) \wedge (\neg B(x) \vee L(x))) & (1),(10), \text{hyp. conjunction} \\
 (12) \quad \forall x H(x) \rightarrow (B(x) \wedge L(x)) & (11), \text{distributive, domination} \\
 (13) \quad \forall x H(x) \rightarrow L(x) & (12), \text{hyp. simplification} \\
 (14) \quad \forall x H(x) \rightarrow (L(x) \wedge (S(x) \vee \neg L(x))) & (7),(13), \text{hyp. conjunction} \\
 (15) \quad \forall x H(x) \rightarrow (L(x) \wedge S(x)) & (14), \text{distributive, domination} \\
 (16) \quad \forall x H(x) \rightarrow S(x) & (15), \text{hyp. simplification}
 \end{array}$$

Here's an alternate proof. Once you create a proof you can often look at it and see a way to make it a little bit shorter and easier to read.

$$\begin{array}{ll}
 (5) \quad \forall x \neg(B(x) \wedge \neg S(x) \wedge L(x)) \iff \forall x \neg[(B(x) \wedge L(x)) \wedge \neg S(x)] & \text{logically equiv to (3)} \\
 (6) \quad \forall x \neg(B(x) \wedge L(x)) \vee S(x) & (5), \text{DeMorgan} \\
 (7) \quad \forall x (B(x) \wedge L(x)) \rightarrow S(x) & (6), \text{implication} \\
 (8) \quad \forall x R(x) \rightarrow \neg(B(x) \wedge \neg L(x)) & \text{contrapositive of (4)} \\
 (9) \quad \forall x R(x) \rightarrow (\neg B(x) \vee L(x)) & (8), \text{DeMorgan} \\
 (10) \quad \forall x H(x) \rightarrow B(x) \wedge (\neg B(x) \vee L(x)) & (1),(9), \text{hyp. conjunction} \\
 (11) \quad \forall x H(x) \rightarrow (B(x) \wedge L(x)) & (10), \text{distributive, domination} \\
 (12) \quad \forall x H(x) \rightarrow S(x) & (7),(11), \text{hyp. syllogism}
 \end{array}$$

Converting this back into English, here's the argument we created. Since no large birds live on honey, we have that any bird that lives on honey must be small. Saying that "Birds that do not live on honey are dull in color" is equivalent to saying that "Anything that is richly colored either is not a bird or it lives on honey". Since we know that hummingbirds are richly colored, then either a hummingbird is not a bird or it lives on honey. However, we know that all hummingbirds are birds and hence it follows that all hummingbirds are birds that live on honey. Finally, since we have already argued that birds that live on honey are small, it follows that hummingbirds are small.