Mirror shape recovery from image curves and intrinsic parameters: Rotationally symmetric and conic mirrors

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Abstract

This paper analyzes the problem of the estimation of the local mirror shape in a catadioptric imaging system. We propose a method to recover the 3D coordinates of mirror surface points as long as there are images of those points, i.e., as long as there is an image of a 3D geometric element that is reflected by those points. For that purpose the information required is the image, the intrinsic parameters of the camera, and the 3D coordinates of 3 points in the scene. The estimation of the local shape can be used to calibrate the system even though that problem is not addressed in this paper. We address the problem of the shape recovery for conic shaped mirrors and rotationally symmetric mirrors. Experimental results for synthetic images are presented.

1. Introduction

Panoramic and omnidirectional images are being increasingly used in many applications. New and interesting applications are being developed. Omnidirectional images are obtained by combining cameras with mirrors. Many of these systems use configurations that assure that the projection is central.

In what concerns the type of mirror employed, rotationally symmetric conic mirrors are usually used. However, those mirrors provide central projection only for special positions of the camera [2, 5, 9]. Suppose that the type of the mirror surface is not known. Can the mirror surface be recovered? Using which information? Previous work on this topic includes some works on reflectance models, polarization, color, photometric characteristics and structured light [1, 6–8, 14, 15, 17]. Other approaches to the problem include using stereo or multiple views [7, 16] and also a moving observer or moving surfaces [10, 12].

In this paper we are interested in non-central projection systems with a perspective camera and a rotationally symmetric mirror. The special case of conic mirrors is also addressed. In this paper we propose a method to recover the mirror surface locally using the following *a priori* information: the image, the intrinsic parameters of the camera and 3 3D points in the scene. In the next section the problem of the mirror surface recovery is addressed and in section 3 we address the initial value problem. This problem results from the fact that the surface reconstruction is obtained from the integration of an ordinary differential equation. In section 4 the validity of the model is demonstrated with experimental results and then we draw the conclusions.

2. Mirror shape recovery

In this section the problem of estimating the 3D mirror points corresponding to the image of a moving point in the scene is addressed. The point can describe any curve in the real scene. The corresponding curve on the image (after the reflection) is tracked. The only *a priori* information required are the image of that very point, the intrinsic parameters of the camera and the 3D coordinates of two or three points in the scene (depending on the model used).

Let $\overrightarrow{x}(s)$ be the point on the mirror surface that we wish to recover. A curve in 3D space will be projected in the image plane and let the curve be parameterized by the variable s which should not be confused with time (see figure 1). It is possible to express $\overrightarrow{x}(s)$ as the sum of its two perpendicular components (see figure 2).

$$\overrightarrow{x}(s) = \overrightarrow{L}(s) + < \overrightarrow{x}(s), \overrightarrow{Vr}(s) > \overrightarrow{Vr}(s) \qquad (1)$$

where $\overrightarrow{Vr}(s)$ is the unitary reflected ray and $< .,. >$ is the inner product. Vector $\overrightarrow{L}(s)$ represents the distance vector from the origin of coordinates to the reflected ray (notice that $< \overrightarrow{L}(s), \overrightarrow{Vr}(s) >= 0$).

Let us now differentiate equation 1 with respect to the

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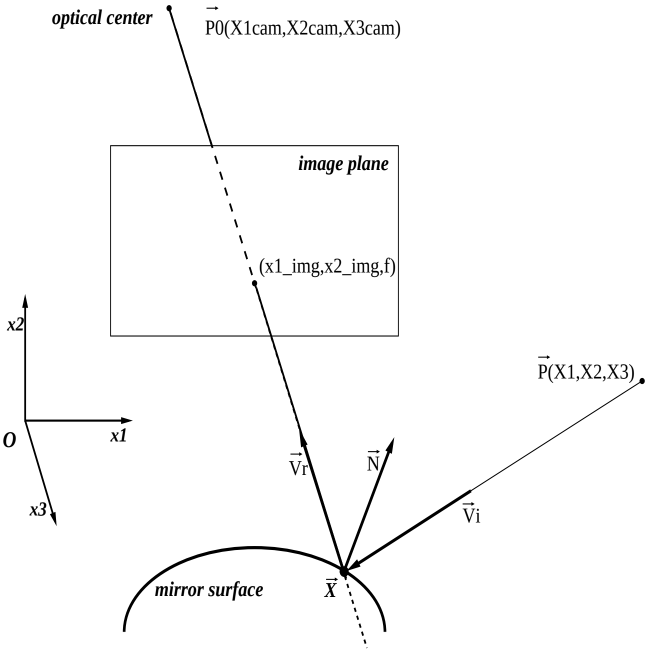

Figure 1: Reflection through a specular mirror

parameter s. It yields:

$$\dot{\vec{x}}(s) = \dot{\vec{L}}(s) + < \vec{x}(s), \vec{Vr}(s) > \dot{\vec{Vr}}(s) + \\ + < \dot{\vec{x}}(s), \vec{Vr}(s) > \vec{Vr}(s) + < \vec{x}(s), \dot{\vec{Vr}}(s) > \vec{Vr}(s) \tag{2}$$

Let us now calculate the inner product of equation 2 with $\dot{\vec{Vr}}(s)$ yielding:

$$< \dot{\vec{x}}(s), \dot{\vec{Vr}}(s) > = < \dot{\vec{L}}(s), \dot{\vec{Vr}}(s) > + \\ + < \vec{x}(s), \vec{Vr}(s) >< \dot{\vec{Vr}}(s), \dot{\vec{Vr}}(s) > \tag{3}$$

Since $\vec{Vr}$ is a unit vector, $< \vec{Vr}, \dot{\vec{Vr}} >= 0$ stands. Rearranging the terms with respect to $< \vec{x}(s), \vec{Vr}(s) >$ and substituting in equation 1 one obtains (the notation $\vec{v}(s)$ is substituted throughout the paper by the shorter form $\vec{v}$):

$$\vec{x} = \frac{< \dot{\vec{x}}, \dot{\vec{Vr}} > \vec{Vr}}{||\dot{\vec{Vr}}||^2} + \vec{L} - \frac{< \dot{\vec{L}}, \dot{\vec{Vr}} > \vec{Vr}}{||\dot{\vec{Vr}}||^2} \tag{4}$$

which is a linear system of differential equations on $\vec{x}$.

Let us now analyze each term of this expression. The reflected ray passes through two known points: the optical center of the camera ($P_0 \equiv (x_{1cam}, x_{2cam}, x_{3cam})$)

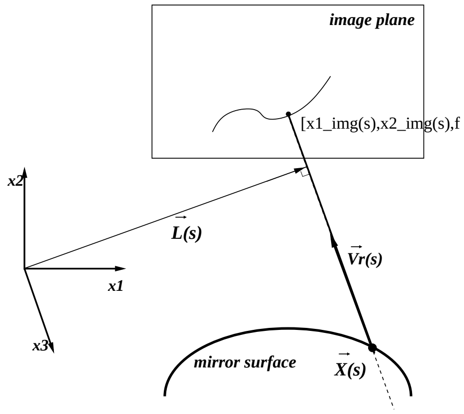

Figure 2: Perpendicular components of the reflected ray.

and the image point ($P_i \equiv (x_{1cam} - x_{1img}, x_{2cam} - x_{2img}, x_{3cam} - f)$), where f is the focal length and (x_{1img}, x_{2img}) are the image coordinates. These expressions however are valid only if there is no rotation between the world coordinate system and the camera coordinate system. This assumption can be considered without loss of generality. The expression of the reflected ray is thus known and its unit vector is given by:

$$\vec{Vr} = \frac{(x_{1img}, x_{2img}, f)^T}{\sqrt{x_{1img}^2 + x_{2img}^2 + f^2}} \tag{5}$$

and its derivative with respect to s is straightforward. Both $\vec{Vr}$ and $\dot{\vec{Vr}}$ depend on the image point and its motion in the image, which can be estimated by tracking a point moving along a curve in the image. Equation 4 is also a function of the vector $\vec{L}$ and its derivative with respect to parameter s - $\dot{\vec{L}}$. Since $\vec{L}$ is the distance vector from the origin of coordinates ($\vec{O}$) to the reflected ray it is straightforward to compute it. The following expression can be used:

$$\vec{L} = \vec{P}_i - \frac{(\vec{P}_i - \vec{O}) \cdot (\vec{P}_0 - \vec{P}_i)}{||\vec{P}_0 - \vec{P}_i||^2} \cdot (\vec{P}_0 - \vec{P}_i) \tag{6}$$

and its derivative is also straightforward. Since for each image point it is possible to know or estimate all coefficients of equation 4 it can be rewritten in matrix form as:

$$A(s)\dot{\vec{x}}(s) = \vec{x}(s) + \varphi(s) \tag{7}$$

where the mass matrix $A(s)$ and the vector $\varphi(s)$ can be easily calculated. Notice however that those entities vary with the parameter s and therefore these differential equations have variable coefficients.

The nature/type of this system of differential equations depends on the matrix $A(s)$. As a matter of fact it will depend on whether $A(s)$ is singular or not. An analysis of the determinant of $A(s)$ shows that it is actually singular and therefore one could be led to think that the system of equations is a DAE (Differential system of Algebraic Equations). However, matrix $A(s)$ is made up of only two linearly independent vectors and it is not possible to transform it into an ODE (Ordinary Differential system of Equations) or a DAE. That means that new restrictions must be considered so that matrix $A(s)$ has rank 3.

Two new restrictions (although one would be enough) can be added to the system by considering the nature of the image projection (perspective projection or orthographic projection if that is the case). The restrictions are $x_{1img} = f(x_1 - x_{1cam})/(x_3 - x_{3cam})$ and $x_{2img} = f(x_2 - x_{2cam})/(x_3 - x_{3cam})$, where x_1, x_2 and x_3 are the coordinates of the 3D point to be recovered and x_{1cam}, x_{2cam} and x_{3cam} are the camera coordinates. Then the system becomes:

$$\begin{cases} a_{11}\dot{x_1} + a_{12}\dot{x_2} + a_{13}\dot{x_3} & = x_1 + k_1 \\ a_{21}\dot{x_1} + a_{22}\dot{x_2} + a_{23}\dot{x_3} & = x_2 + k_2 \\ a_{31}\dot{x_1} + a_{32}\dot{x_2} + a_{33}\dot{x_3} & = x_3 + k_3 \\ x_{3cam}x_{1img} - fx_{1cam} & = -fx_1 + x_{1img}x_3 \\ x_{3cam}x_{2img} - fx_{2cam} & = -fx_2 + x_{2img}x_3 \end{cases} \tag{8}$$

This equation is a DAE of index 1. The index of a DAE is the minimum number of derivatives that have to be taken on some of the equations so that an explicit ODE can be obtained. So, taking the derivatives of the new restrictions with respect to the parameter s the resulting ODE system is obtained:

$$\begin{cases} a_{11}\dot{x_1} + a_{12}\dot{x_2} + a_{13}\dot{x_3} & = x_1 + k_1 \\ a_{21}\dot{x_1} + a_{22}\dot{x_2} + a_{23}\dot{x_3} & = x_2 + k_2 \\ a_{31}\dot{x_1} + a_{32}\dot{x_2} + a_{33}\dot{x_3} & = x_3 + k_3 \\ f\dot{x_1} - x_{1img}\dot{x_3} & = v_{x_1}x_3 - v_{x_1}x_{3cam} \\ f\dot{x_2} - x_{2img}\dot{x_3} & = v_{x_2}x_3 - v_{x_2}x_{3cam} \end{cases} \tag{9}$$

or in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ f & 0 & -x_{1img} \\ 0 & f & -x_{2img} \end{bmatrix} \dot{\vec{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & v_{x_1} \\ 0 & 0 & v_{x_2} \end{bmatrix} \vec{x} + \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{bmatrix}$$

$$\Longleftrightarrow A(s)\dot{\vec{x}} = B(s)\vec{x} + \vec{\varphi}(s) \tag{10}$$

where $k_4 = -v_{x_1}x_{3cam}$ and $k_5 = -v_{x_2}x_{3cam}$.

Since the new restrictions introduce a third linearly independent condition, the rank of matrix $A(s)$ is 3. Furthermore, matrix $A(s)$ is now over-determined and so its pseudo-inverse matrix can be used to estimate $\dot{\vec{x}}(s)$. The final system becomes:

$$\dot{\vec{x}}(s) = (A^TA)^{-1}A^TB\vec{x}(s) + (A^TA)^{-1}A^T\vec{\varphi} \tag{11}$$

which an ODE system with variable coefficient matrices.

3. Initial Value Problem

The initial value problem is still unsolved. Since the main interest of this method is the recovery of the 3D shape of the reflecting mirror with minimal *a priori* data, it is important that the initial value problem be solved with minimal initial knowledge of the system. Let us now present a possible solution for this problem.

3.1. Knowing the Starting Point

Solving equation 11 requires that the initial value $\vec{x}_0 = \vec{x}(0)$ is known. Consider that one line (or any arbitrary curve) is tracked in the image. Equation 11 can be used to estimate the 3D coordinates of the mirror points corresponding to the points being tracked in the image, as long as the 3D coordinates of the initial mirror point are known.

3.2. Closed Contours

Assume that the scene contains some closed contours such as rectangles, triangles, circles or any arbitrary closed contour (examples of this are doors or chess board-like pavements). The interest of those contours lies on the fact that equation 11 can be used to estimate the 3D coordinates of the mirror points that reflect the contours without the knowledge of an initial point. As a matter of fact all that is required is that the contour is traversed by iterating the method until the starting point coincides with the final point (once the figure is closed). If one has a good initial guess, the method should converge rapidly using well-known methods for solving non-linear equations.

3.3. Conic reflectors

The previous solution is interesting but as we shall see, experimental results show that the convergence of such method is usually poor unless good initial estimates are known. Since the many of the mirrors used in omnidirectional vision correspond to conics, this can be used to obtain an initial guess. Consider then that the reflector mirror corresponds to a conic (including elliptic, parabolic, hyperbolic and spherical). The equation for such mirrors is:

$$x_1^2 + x_2^2 + Ax_3^2 + Bx_3 = C \tag{12}$$

or rewriting:

$$x_1^2 + x_2^2 + A\left(x_3 + \frac{B}{2A}\right)^2 = \frac{B^2}{4A} + C \tag{13}$$

where this equation assumes that the conic is centered in the origin of the coordinate system and aligned with the coordinate axes. The general conic equation with an arbitrary orientation can be used although it increases the number of unknowns. This problem is being currently addressed.

The generic mirror point is then given by:

$$\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{\frac{C+\frac{B^2}{4A}-x_1^2-x_2^2}{A}} - \frac{B}{2A} \end{bmatrix} \tag{14}$$

Taking the partial derivatives with respect to the spatial coordinates x_1 and x_2 it yields:

$$\begin{cases} \frac{\partial \overrightarrow{x}}{\partial x_1} = \begin{bmatrix} 1 \\ 0 \\ -\frac{x_1}{\sqrt{\frac{C+\frac{B^2}{4A}-x_1^2-x_2^2}{A}} - \frac{B}{2A}} \end{bmatrix} \\ \frac{\partial \overrightarrow{x}}{\partial x_2} = \begin{bmatrix} 0 \\ 1 \\ -\frac{x_2}{\sqrt{\frac{C+\frac{B^2}{4A}-x_1^2-x_2^2}{A}} - \frac{B}{2A}} \end{bmatrix} \end{cases} \tag{15}$$

and the normal vector to the mirror surface becomes:

$$\overrightarrow{N} = \frac{\frac{\partial \overrightarrow{x}}{\partial x_1} \times \frac{\partial \overrightarrow{x}}{\partial x_2}}{\left|\left|\frac{\partial \overrightarrow{x}}{\partial x_1} \times \frac{\partial \overrightarrow{x}}{\partial x_2}\right|\right|} \tag{16}$$

Furthermore, the incident ray $\overrightarrow{Vi}$ can be recovered using the normal vector and the reflected ray $\overrightarrow{Vr}$:

$$\overrightarrow{V_i} = \overrightarrow{V_r} - 2 < \overrightarrow{N}, \overrightarrow{V_r} > \overrightarrow{N} \tag{17}$$

where $\overrightarrow{Vr}$ can be calculated using equation 5 and $\overrightarrow{Vi} = \alpha(\overrightarrow{x} - \overrightarrow{P})$, being $\overrightarrow{P}$ the vector with the 3D world coordinates of the scene point (see figure 1). If the 3D coordinates of a point are considered to be known as well as the reflected ray $\overrightarrow{Vr}$, equation 17 provides 3 equations for the following seven unknowns: x_1, x_2, x_3, A, B, C and α. However, by using the perspective projection equations the number of unknowns can be decreased by two since $x_1 = x_{1img}(x_3 - x_{3cam})/f + x_{1cam}$ and $x_2 = x_{2img}(x_3 - x_{3cam})/f + x_{2cam}$. The system of equations then includes 3 equations for five unknowns: x_3, A, B, C and α. We still have more unknowns than equations. Each additional 3D point adds 3 equations and two new unknowns:x_{3j} and α_j. Therefore the minimal number of 3D points is three since in that case there will be 9 equations with 9 unknowns (the shape parameters are the same). The knowledge of the 3D coordinates of three points in the scene and of their corresponding image coordinates allows the computation of an estimate of the initial value required by equation 11. Additionally estimates for the conic shape parameters A, B and C are also obtained. These values are obtained as solutions of the nonlinear system of equations.

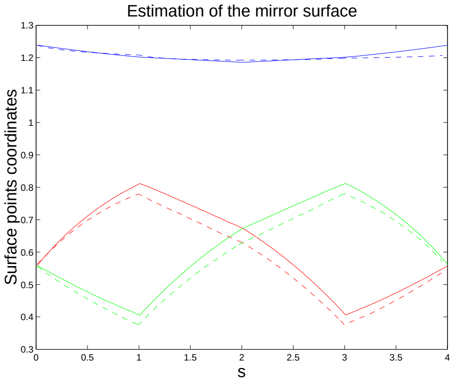

Figure 3: Estimated values (dashed lines) and true values (solid lines) of the mirror surface. There are three pairs of lines in the graphic: the upper is the x_3 coordinate and the two in the bottom of the plot are the x_1 and x_2 coordinates. Elliptic mirror shape.

4. Experimental results

In this section experimental results obtained with synthetic images are presented.

For this set of experimental results an elliptic mirror and a perspective camera with a focal length of $5mm$ were considered. We also performed test with an hyperbolic mirror. Using equation 17 the images of several geometrical elements (lines, rectangles and curves) were obtained. The ground truth values computed were the image coordinates ($x_{1img}(s)$ and $x_{2img}(s)$), the image flow ($v_{x_1}(s)$ and $v_{x_2}(s)$), the coordinates of the corresponding point on the mirror surface ($\overrightarrow{x}(s)$, to be recovered) and the 3D coordinates of the points in the scene.

The first experimental results correspond to the estimation of the coordinates of mirror surface points obtained by using a 3D square. The results correspond to the solution of equation 11 with known initial value, i.e., when $\overrightarrow{x}_0$ is known. Figures 3 and 4 show the estimated values and their relative errors. The errors are small for all the coordinates. Figure 5 displays in 3D the ground truth points and the points recovered on the mirror surface.

As it can be seen the recovered curve is not closed. To find out a good starting point the approach described in sub-

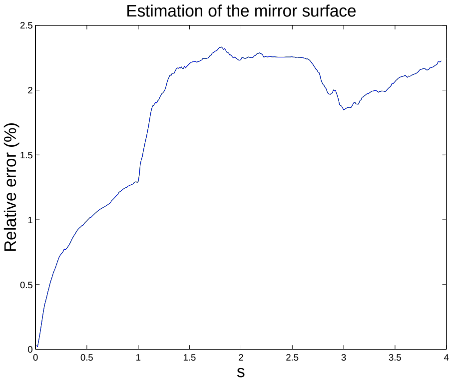

Figure 4: Relative error in the estimation of the points on the mirror surface.

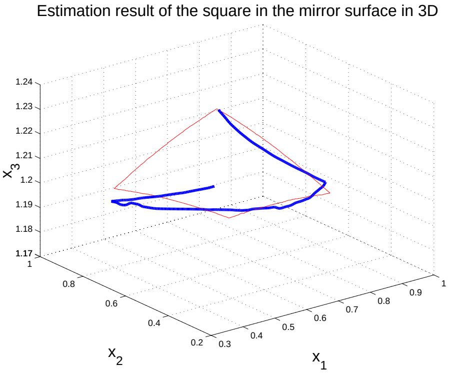

Figure 5: 3D representation of the curve recovered (thicker line) and the ground truth curve.

section 3.2 was used. As a result of the nonlinear iterative matching process a closed curve was recovered. The results are plotted in figure 6.

Finally, the restriction of the conic mirror was used to estimate the initial value. In this test the curve used was a 3D line segment line instead of a closed curve. Figures 7 and 8 show respectively the comparison of the estimated and ground truth values for each spatial coordinate and the relative error in the estimation the curve recovered.

We also performed tests with an hyperbolic mirror. Figure 9 shows that also in hyperbolic the estimation results are good.

5. Summary and Conclusions

In this paper the problem of the estimation the mirror shape of an omnidirectional vision system is addressed. A method to locally recover the 3D coordinates of the mirror points that reflect a 3D curve into the image plane is proposed.

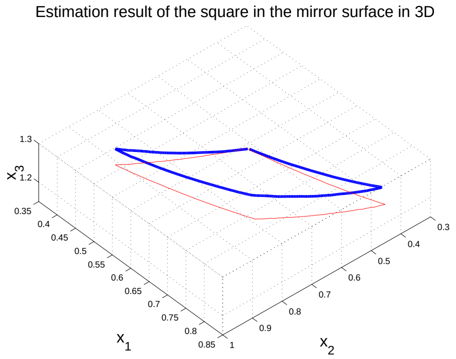

Figure 6: 3D space representation of the curve recovered (thicker line) and the ground truth curve, after the iterative matching of the initial and final points.

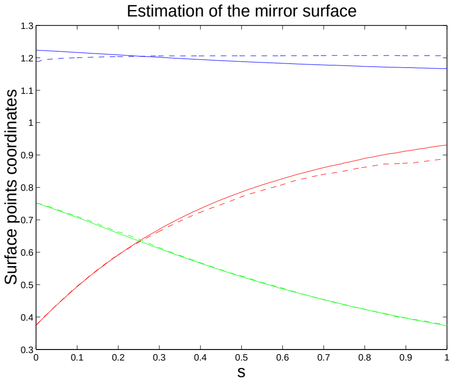

Figure 7: Estimated values (dashed lines) and true values (solid lines) of the mirror surface. There are three pairs of lines in the graphic: the upper is the x_3 coordinate and the two in the bottom of the plot are the x_1 and x_2 coordinates. Conic mirror shape was assumed.

The estimation of the local shape can be used to calibrate the omnidirectional system. If the curve in the image plane is traversed the problem of estimating the mirror points is solved by a system of differential equations, once the initial value is available. However a good initial value is required and that is the main difficulty. If the mirror is considered to be conic (including elliptic, parabolic, hyperbolic and spherical) then the 3D coordinates of three points in the scene can be used to get a good initial estimate. The experimental results show that the relative error in the estimation of the 3D coordinates of the mirror points is small. In the future several extensions are being considered namely to other types of mirrors and to study the methods numerical stability.

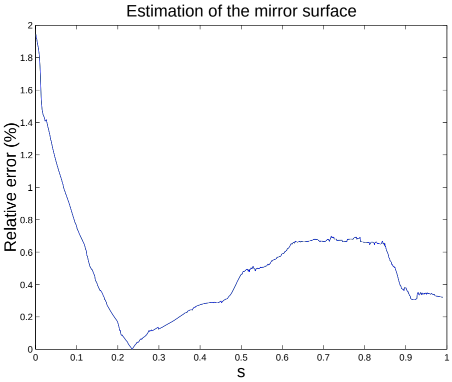

Figure 8: Relative error in the estimation of the line segment points on the mirror surface. Conic mirror shape was assumed.

Acknowledgments

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References

[1] L. Weiss A. Sanderson and S. Nayar. Structured highlight inspection of specular surfaces. *PAMI*, 10(1):44–55, January 1988.

[2] S. Baker and S. Nayar. A theory of catadioptric image formation. In *IEEE ICCV*, pages 35–42, Bombay, 1998.

[3] Max Born and Emil Wolf. *Principles of Optics*. Pergamon Press, 1965.

[4] Donald Burkhard and David Shealy. Flux density for ray propagation in geometrical optics. *J. Optical Soc. of America*, 63(3):299–304, March 1973.

[5] C. Geyer and K. Daniilidis. A unifying theory for central panoramic and practical implications. In *ECCV*, pages 445–461, Dublin, 2000.

[6] G. Healey and T. Binford. Local shape from specularity. *CVGIP*, 42(1):62–86, April 1988.

[7] K. Ikeuchi. Determinig surface orientations of specular surfaces by using the photometric stereo method. *PAMI*, 3(6):661–669, November 1981.

[8] J. Koenderink and A. Doorn. Photometric invariants related to solid shape. *Optica Acta*, 27(7):981–996, 1980.

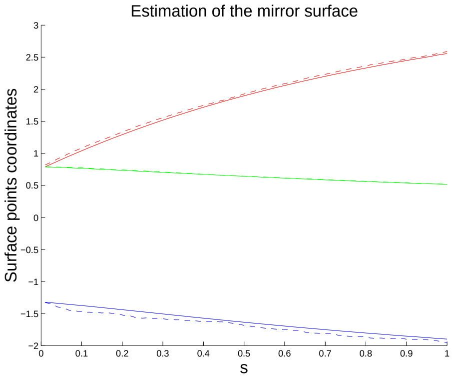

Figure 9: Estimated values (dashed lines) and true values (solid lines) of the mirror surface. There are three pairs of lines in the graphic. From the top to bottom: the upper is the x_1 coordinate, the middle pair is x_2 and in the bottom the pair of lines corresponds to x_3 coordinates. Hyperbolic mirror shape.

[9] S. Lin and R. Bajcsy. True single view point cone mirror omni-directional catadioptric system. In *IEEE ICCV*, Vancouver, July 2001.

[10] M. Longuet-Higgins. Reflection and refraction at a random moving surface: i, ii and iii. *J. Optical Soc. of America*, 50(9):838–856, September 1960.

[11] Shree Nayar and Simon Baker. Catadioptric image formation. In *DARPA Image Understanding Workshop*, New Orleans, May 1997.

[12] Michael Oren and Shree Nayar. A theory of specular surface geometry. *IJCV*, 1996.

[13] Michael Grossberg Rahul Swaminathan and Shree Nayar. Caustics of catadioptric cameras. In *ICCV'01*, Vancouver, Canada, July 2001. IEEE.

[14] K. Ikeuchi S. Nayar and T. Kanade. Determinig shape and reflectance of hybrid surface by photometric sampling. *IEEE Transactions on Robotics and Automation*, 6(4):418–431, August 1990.

[15] X. Fang S. Nayar and T. Boult. Separation of reflection components using color and polarization. *IJCV*, 21(3):163–186, February 1997.

[16] H. Schultz. Retrieving shape information from multiple images of a specular surface. *PAMI*, 16(2):195–200, February 1994.

[17] K. Torrence and E. Sparrow. Theory for off-specular reflection from roughened surfaces. *J. Optical Soc. of America*, 57(9):1105–1114, September 1967.