CSE 441T/541T Advanced Algorithms **November 5, 2008**

Homework 4 Solutions

Note: there are many correct reductions that you can use to prove NP-completeness, and there may be multiple ways to prove the approximation bounds in this assignment. These solutions are a combination of my own ideas and the best approaches I've heard from you.

1. *(15 pts) In class, we showed that INDEPENDENT-SET $\leq_p$ VERTEX-COVER. Recall that a graph G with n vertices has an independent set of size at least k iff it has a vertex cover of size at most $n-k$; in particular, the vertices* not *in the cover form an independent set.*

   In class, we saw a 2-approximation algorithm A for finding the minimal vertex cover. Professor Ptolemy suggests that one can derive a constant-factor approximation algorithm for independent set by first reducing it to a vertex cover problem as above, then applying algorithm A. Is the professor correct? Justify your answer, either by proving that there exists a constant c for which the above approach gives a c-approximation, or by showing that for any constant c, there exist graphs G (with arbitrarily many vertices) for which the algorithm is not *a c approximation.*

   The professor's algorithm is *not* a constant-factor approximation. To see this, consider a graph G constructed as follows. G consists of $n-1$ pairs of vertices (u_i, v_i), each connected by an edge, plus a single additional vertex w with no incident edges.

   Graph G contains the independent set $\{u_1, \ldots u_{n-1}, w\}$, of total size n. However, our algorithm A for vertex cover will pick *every* edge of G and add both its endpoints to the cover. A therefore produces a cover of size $2(n-1)$, leaving the singleton set $\{w\}$ as the corresponding independent set. The professor's algorithm can therefore yield an approximation ratio as bad as n for a graph of size $2n+1$; hence, it is *not* a constant-factor approximation.

2. *(10 pts) Consider the following problem. You are given a simple undirected weighted graph G and an integer $f \leq |V|$. Your goal is to pick the location for f vertices of G as casino locations so that the* length of the shortest path *between any vertex in G and its closest casino is the minimum possible. From this optimization problem we create the following decision problem that we will call the CASINO-PLACEMENT problem:*

   Given an undirected weighted graph G, an integer $f \leq |V|$, and a real number $d \geq 0$, is it possible to select f vertices of G as casino locations so that the length of the shortest path between any vertex in G and its nearest casino is at most d.

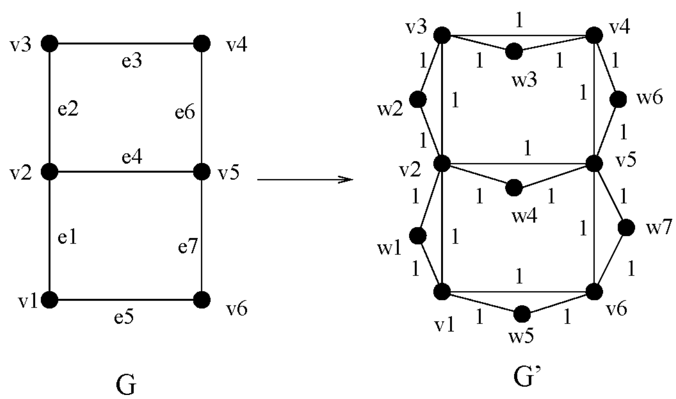
   Either Prove that there is a polynomial time algorithm for the CASINO-PLACEMENT problem, Or prove that CASINO-PLACEMENT is NP-complete.

   HERE'S THE SOLUTION. This problem is more commonly knows as the "FIRE-STATION" problem, because having a fire-station nearby serves a more compelling public good than having a Casino nearby).

   We prove that FIRE-STATION-PLACEMENT is NP-complete. To see that it is NP we use the set of vertices for placing the fire stations as a certificate. Clearly, there is a polynomial time verification algorithm.

We now prove that VERTEX-COVER $\leq_p$ FIRE-STATION-PLACEMENT. Let $\langle G, k \rangle$ be the input for VERTEX-COVER. Without loss of generality, we assume that no vertices in G have degree 0. (If G had any vertices of degree 0, then remove them. Notice that this won't affect the size of a minimum vertex cover of G.)

The graph for the fire-station-placement problem $G' = (V', E')$ is obtained from $G = (V, E)$ as follows. Let $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. We will create a new vertex w_i for each edge in E. Let $W = \{w_1, \ldots, w_m\}$. Then $V' = V \cup W$ and for each edge $e = (u, v) \in E$ (with associated vertex w) we place the following 3 edges in E': $(u, v), (u, w), (v, w)$. All edges have a weight of 1. So $|V'| = |V| + |E|$ and $|E'| = 3|E|$. Here is an example of this transformation:



Clearly G' can be constructed in polynomial time. The input for FIRE-STATION-PLACEMENT is $\langle G', f = k, d = 1 \rangle$.

We now prove that G has a vertex cover of size k if and only if G' has a placement of k fire stations so that each vertex has distance of at most 1 to its nearest fire station. Suppose that G has a vertex cover C of size k. We show that by placing a firestation in G' at each vertex in C all vertices are directly connected to a firestation (or have a firestation). Suppose not. Let u be such a vertex greater than distance 1 from its nearest fire station. First suppose that $u \in V$ (i.e. an original vertex from G). Since all vertices in G have at least one edge, u must be connected in G' to some vertex $v \in V$ where v is also not in C. This contradicts that C is a vertex cover. Next suppose that $u \in W$ (it is one of the added vertices), then u is connected to two vertices v_1 and v_2 where $(v_1, v_2) \in E$ and there is not a firestation at v_1 or v_2 thus contradicting that C was a vertex cover.

We first prove that if F is the set of vertices to place fire stations so that $|F| = k$ and all vertices have distance at most 1 to some fire station, then there is a vertex cover of size k in G.

We first transform F into a fire station placement solution F' in which $|F'| = |F|$ and in which $F' \subseteq V$ (i.e. F' only includes original vertices). We build F' from F by doing the following for each vertex $w \in W \cap F$ (in any order). Suppose w is the vertex added corresponding to edge (u_1, u_2). In this case, we can simply move the fire station at vertex w to u_1 (or anywhere else if u_1 already has a fire station). Notice that the fire station at w could only be a nearest fire station for w, u_1 and u_2 and hence if there is a fire station at u_1 then each of w, u_1 and u_2 can use u_1 as a nearest firestation. Hence we maintain the invariant (started with F) that our solution at each step (and hence F') satisfies the requirements that each vertex in

$V' = V \cup W$ is within distance 1 from a fire station.

We now prove that F' is a vertex cover in G. Suppose not. Then there is an edge $(u, v) \in G$ such that $u \notin F'$ and $v \notin F'$. Consider the vertex w corresponding to the edge (u, v) (i.e the triangle u,v,w). Notice that w is connected only to u and v and thus w must have a distance of greater than 1 to its nearest fire station thus contradicting the requirements for a proper firestation placement in G'.